Chapter 3

The Derivative

In developing the S-I-R model in chapter 1 we took the idea of the rate of change of a population as intuitively clear. The rate at which one quantity changes with respect to another is a central concept of calculus and leads to a broad range of insights. The chief purpose of this chapter is to develop a fuller understanding—both analytic and geometric—of the connection between a function and its rate of change. To do this we will introduce the concept of the **derivative** of a function.

3.1 Rates of Change

The Changing Time of Sunrise

The time of sunrise is a function

The sun rises at different times, depending on the date and location. At 40° N latitude (New York, Beijing, and Madrid are about at this latitude) in the year 1990, for instance, the sun rose at

7:16	on January 23,
5:58	on March 24,
4:52	on July 25.

Clearly the time of sunrise is a function of the date. If we represent the time of sunrise by T (in hours and minutes) and the date by d (the day of the year), we can express this functional relation in the form $T = T(d)$. For example, from the table above we find $T(23) = $ 7:16. It is not obvious from the table, but it is also true that the the rate at which the time of sunrise is changing is different at different times of the year—T' varies as d varies.

We can see how the rate varies by looking at some further data for sunrise at the same latitude, taken from *The Nautical Almanac for the Year 1990*:

date	**time**	**date**	**time**	**date**	**time**
January 20	7:18	March 21	6:02	July 22	4:49
23	7:16	24	5:58	25	4:52
26	7:14	27	5:53	28	4:54

Calculate the rate using earlier and later dates

Let's use this table to estimate the rate at which the time of sunrise is changing on January 23. We'll use the times three days earlier and three days later, and compare them. On January 26 the sun rose 4 minutes earlier than on January 20. This is a change of −4 minutes in 6 days, so the rate of change is

$$-4\ \frac{\text{minutes}}{6\ \text{days}} \approx -.67 \text{ minutes per day.}$$

We say this is the **rate** at which sunrise is changing on January 23, and we write

$$T'(23) \approx -.67\ \frac{\text{minutes}}{\text{day}}.$$

The rate is negative because the time of sunrise is decreasing—the sun is rising earlier each day.

Similarly, we find that around March 24 the time of sunrise is changing approximately $-9/6 = -1.5$ minutes per day, and around July 25 the rate is $5/6 \approx +.8$ minutes per day. The last value is positive, since the time of sunrise is increasing—the sun is rising later each day in July. Since March 24 is the 83rd day of the year and July 25 is the 206th, using our notation for rate of change we can write

$$T'(83) \approx -1.5\ \frac{\text{minutes}}{\text{day}}; \qquad T'(206) \approx .8\ \frac{\text{minutes}}{\text{day}}.$$

Notice that, in each case, we have calculated the rate on a given day by using times *shortly before* and *shortly after* that day. We will continue this pattern wherever possible. In particular, you should follow it when you do the exercises about a falling object, at the end of the section.

Once we have the rates, we can estimate the time of sunrise for dates not given in the table. For instance, January 28 is five days after January 23, so the total change in the time of sunrise from January 23 to January 28 should be approximately

$$\Delta T \approx -.67\ \frac{\text{minutes}}{\text{day}} \times 5 \text{ days} = -3.35 \text{ minutes.}$$

In whole numbers, then, the sun rose 3 minutes earlier on January 28 than on January 23. Since sunrise was at 7:16 on the 23rd, it was at 7:13 on the 28th.

By letting the change in the number of days be negative, we can use this same reasoning to tell us the time of sunrise on days shortly *before* the given dates. For example, March 18 is −6 days away from March 24, so the change in the time of sunrise should be

$$\Delta T \approx -1.5\ \frac{\text{minutes}}{\text{day}} \times -6 \text{ days} = +9 \text{ minutes}.$$

Therefore, we can estimate that sunrise occurred at 5:58 + 0:09 = 6:07 on March 18.

Changing Rates

Suppose instead of using the tabulated values for March we tried to use our January data to *predict* the time of sunrise in March. Now March 24 is 60 days after January 23, so the change in the time of sunrise should be approximately

$$\Delta T \approx -.67\ \frac{\text{minutes}}{\text{day}} \times 60 \text{ days} = -40.2 \text{ minutes},$$

and we would conclude that sunrise on March 24 should be at about 7:16 − 0:40 = 6:37, which is more than half an hour later than the actual time! This is a problem we met often in estimating future values in the S-I-R model. implicitly assume that the time of sunrise changes at the fixed rate of −.67 minutes per day over the entire 60-day time-span. But this turns out not to be true: the rate actually varies, and the variation is too great for us to get a useful estimate. Only with a much smaller time-span does the rate not vary too much.

Predictions over long time spans are less reliable

Here is the same lesson in another context. Suppose you are travelling in a car along a busy road at rush hour and notice that you are going 50 miles per hour. You would be fairly confident that in the next 30 seconds (1/120 of an hour) you will travel about

$$\Delta \text{ position} \approx 50\ \frac{\text{miles}}{\text{hour}} \times \frac{1}{120} \text{ hour} = \frac{5}{12} \text{ mile} = 2200 \text{ feet}.$$

The actual value ought to be within 50 feet of this, making the estimate accurate to within about 2% or 3%. On the other hand, if you wanted to estimate how far you would go in the next 30 minutes, your speed would probably fluctuate too much for the calculation

$$\Delta\,\text{position} \approx 50\ \frac{\text{miles}}{\text{hour}} \times \frac{1}{2}\ \text{hours} = 25\ \text{miles}$$

to have the same level of reliability.

Other Rates, Other Units

In the S-I-R model the rates we analyzed were **population growth rates**. They told us how the three populations changed over time, in units of persons per day. If we were studying the growth of a colony of mold, measuring its size by its weight (in grams), we could describe *its* population growth rate in units of grams per hour. In discussing the motion of an automobile, the rate we consider is the **velocity** (in miles per hour), which tells us how the distance from some starting point changes over time. We also pay attention to the rate at which *velocity* changes over time. This is called **acceleration**, and can be measured in miles per hour per hour.

Examples of rates

While many rates do involve changes with respect to time, other rates do not. Two examples are the survival rate for a disease (survivors per thousand infected persons) and the dose rate for a medicine (milligrams per pound of body weight). Other common rates are the annual birth rate and the annual death rate, which might have values like 19.3 live births per 1,000 population and 12.4 deaths per 1,000 population. Any quantity expressed as a percentage, such as an interest rate or an unemployment rate, is a rate of a similar sort. An unemployment rate of 5%, for instance, means 5 unemployed workers per 100 workers. There are many other examples of rates in the economic world that make use of a variety of units—exchange rates (e.g., francs per dollar), marginal return (e.g., dollars of profit per dollar of change in price).

The rate of change of a rate

Sometimes we even want to know the rate of change of one rate with respect to another rate. For example, automobile fuel economy (in miles per gallon—the first rate) changes with speed (in miles per hour—the second rate), and we can measure the rate of change of fuel economy with speed. Take a car that goes 22 miles per gallon of fuel at 50 miles per hour, but only 19 miles per gallon at 60 miles per hour. Then its fuel economy is changing approximately at the rate

$$\frac{\Delta\,\text{fuel economy}}{\Delta\,\text{speed}} = \frac{19 - 22 \text{ miles per gallon}}{60 - 50 \text{ miles per hour}}$$
$$= -.3 \text{ miles per gallon } \textit{per} \text{ mile per hour.}$$

Exercises

A falling object. These questions deal with an object that is held motionless 10,000 feet above the surface of the ocean and then dropped. Start a clock ticking the moment it is dropped, and let D be the number of feet it has fallen after the clock has run t seconds. The following table shows some of the values of t and D.

time (seconds)	distance (feet)
0	0.00
1	15.07
2	56.90
3	121.03
4	203.76
5	302.00
6	413.16
7	535.10

1. What units do you use to measure velocity—that is, the rate of change of distance with respect to time—in this problem?

2. a) Make a careful graph that shows these eight data points. Put *time* on the horizontal axis. Label the axes and indicate the units you are using on each.
b) The slope of any line drawn on this time–distance graph has the units of a *velocity*. Explain why.

3. Make three estimates of the velocity of the falling object at the 2 second mark using the distances fallen between these times:
   i) from 1 second to 2 seconds;
   ii) from 2 seconds to 3 seconds;
   iii) from 1 second to 3 seconds.

4. a) Each of the estimates in the previous question corresponds to the slope of a particular line you can draw in your graph. Draw those lines and label each with the corresponding velocity.

b) Which of the three estimates in the previous question do you think is best? Explain your choice.

5. Using your best method, estimate the velocity of the falling object after 4 seconds have passed.

6. Is the object speeding up or slowing down as it falls? How can you tell?

7. Approximate the velocity of the falling object after 7 seconds have passed. Use your answer to estimate the number of feet the object has fallen after 8 seconds have passed. Do you think your estimate is too high or too low? Why?

8. For those of you attending a school at which you pay tuition, find out what the tuition has been for each of the last four years at your school.

a) At what rate has the tuition changed in each of the last three years? What are the units? In which year was the rate the greatest?

b) A more informative rate is often the **inflation rate** which doesn't look at the dollar change per year, but at the percentage change per year—the dollar change in tuition in a year's time expressed as a percentage of the tuition at the beginning of the year. What is the tuition inflation rate at your school for each of the last three years? How do these rates compare with the rates you found in part (a)?

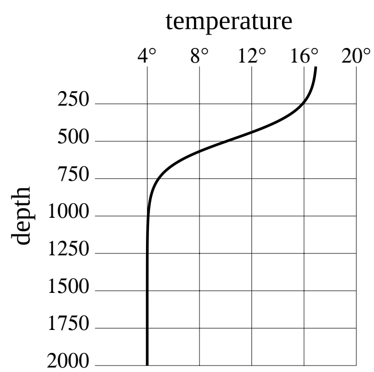
c) If you were interested in seeing how the inflation rate was changing over time, you would be looking at the rate of change of the inflation rate. What would the units of this rate be? What is the rate of change of the inflation rate at your school for the last two years?

d) Using all this information, what would be your estimate for next year's tuition?

9. Your library should have several reference books giving annual statistics of various sorts. A good one is the *Statistical Yearbook* put out by the United Nations with detailed data from all over the world on manufacturing, transportation, energy, agriculture, tourism, and culture. Another is *Historical Statistics of the United States, Colonial Times–1970*. Select an interesting

quantity and compare its growth rate at different times or for different countries. Calculate this growth rate over a stretch of four or five years and report whether there are any apparent patterns. Calculate the rate of change of this growth rate and interpret its values.

10. Oceanographers are very interested in the **temperature profile** of the part of the ocean they are studying. That is, how does the temperature T (in degrees Celsius) vary with the depth d (measured in meters). A typical temperature profile might look something like the following:



(Note that since water is densest at 4° C, water at that temperature settles to the bottom.)

a) What are the units for the rate of change of temperature with depth?

b) In this graph will the rate be positive or negative? Justify your answer.

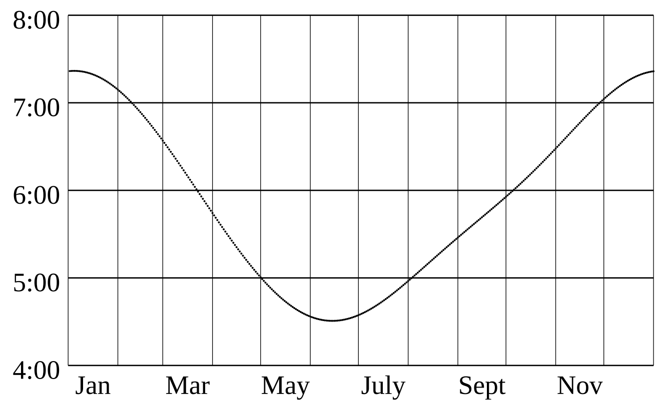
c) At what rate is the temperature changing at a depth of 0 meters? 500 meters? 1000 meters?

d) Sketch a possible temperature profile for this location if the surface is iced over.

e) This graph is not oriented the way our graphs have been up til now. Why do you suppose oceanographers (and geologists and atmospheric scientists) often draw graphs with axes positioned like this?

3.2 Microscopes and Local Linearity

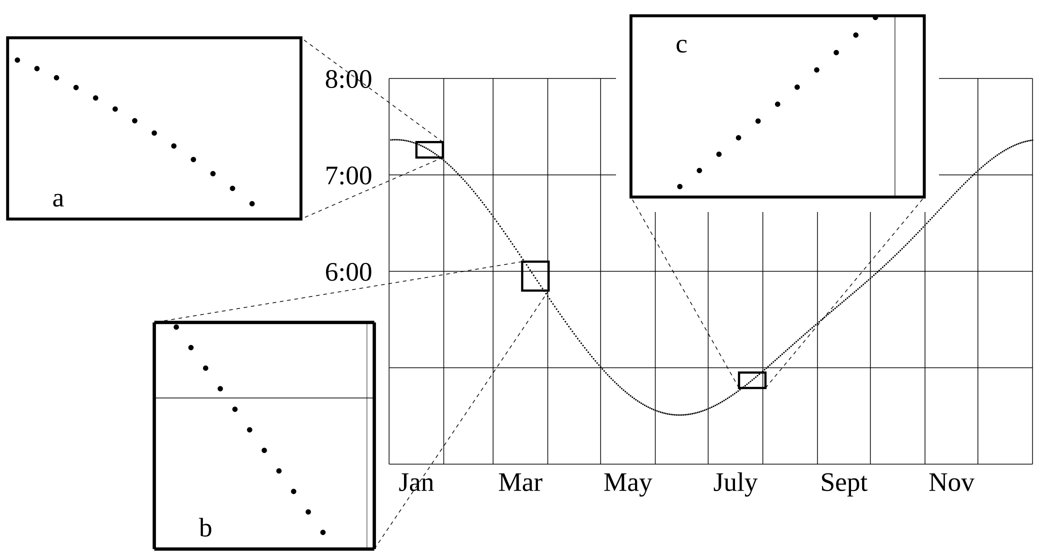
The Graph of Data



This section is about seeing rates geometrically. We know from chapter 1 that we can visualize the rate of change of a *linear* function as the slope of its graph. Can we say the same thing about the sunrise function? The graph of this function appears at the right; it plots the time of sunrise (over the course of a year at 40° N latitude), as a function of the date. The graph is curved, so the sunrise function is not linear. There is no immediately obvious connection between rate and slope. In fact, it isn't even clear what we might mean by the *slope* of this graph! We can make it clear by using a **microscope**.

Zoom in on the graph with a microscope

Imagine we have a microscope that allows us to "zoom in" on the graph near each of the the three dates we considered in section 1. If we put each magnified image in a window, then we get the following:



Notice how different the graph looks under the microscope. First of all, it now shows up clearly as a collection of separate points—one for each day of the year. Second, the points in a particular window lie on a line that is essentially straight. The straight lines in the three windows have very different slopes, but that is only to be expected.

The graph looks straight under a microscope

What is the connection between these slopes and the rates of change we calculated in the last section? To decide, we should calculate the slope in each window. This involves choosing a pair of points (d_1, T_1) and (d_2, T_2) on the graph and calculating the ratio

$$\frac{\Delta T}{\Delta d} = \frac{T_2 - T_1}{d_2 - d_1}.$$

In window **a** we'll take the two points that lie three days on either side of the central date, January 23. These points have coordinates $(20, 7{:}18)$ and $(26, 7{:}14)$ (table, page 102). The slope is thus

$$\frac{\Delta T}{\Delta d} = \frac{7{:}14 - 7{:}18}{26 - 20} = \frac{-4 \text{ minutes}}{6 \text{ days}} = -.67\,\frac{\text{minutes}}{\text{day}}.$$

If we use the same approach in the other two windows we find that the line in window **b** has slope -1.5 min/day, while the line in window **c** has slope $+.8$ min/day. These are exactly the same calculations we did in section 1 to determine the rate of change of the time of sunrise around January 23, March 24, and July 25, and they produce the same values we obtained there:

Slope and rate calculations are the same

$$T'(23) \approx -.67\,\frac{\text{min}}{\text{day}}, \qquad T'(83) \approx -1.5\,\frac{\text{min}}{\text{day}}, \qquad T'(206) \approx .8\,\frac{\text{min}}{\text{day}}.$$

This is a crucial observation which we use repeatedly in other contexts; let's pause and state it in general terms:

> The **rate of change of a function** at a point is equal to the **slope of its graph** at that point, if the graph looks straight when we view it under a microscope.

The Graph of a Formula

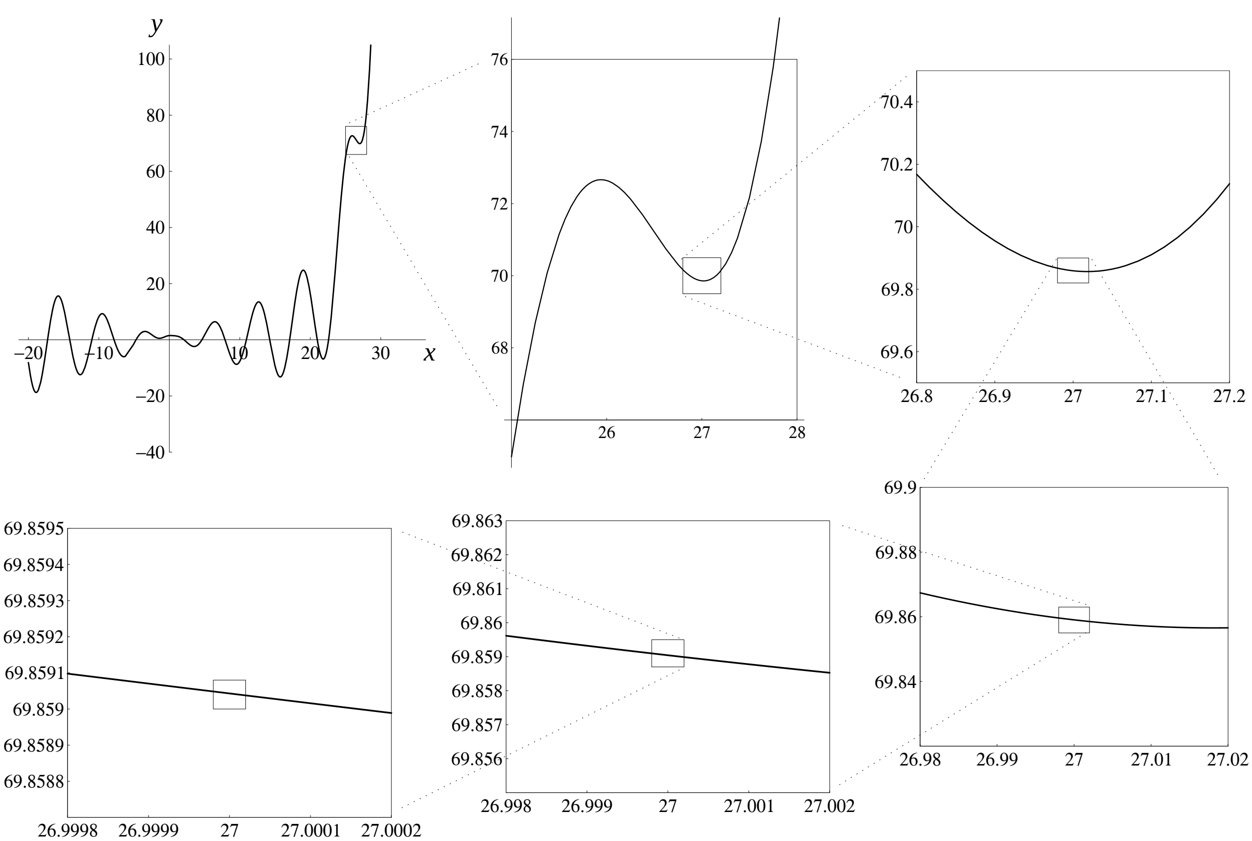
Rates and slopes are really the same thing—that's what we learn by using a microscope to view the graph of the sunrise function. But the sunrise graph consists of a finite number of disconnected points—a very common situation when we deal with data. In such cases it doesn't make sense to magnify the graph too much. For instance, we would get no useful information from a window that was narrower than the space between the data points. There is no such limitation if we use a microscope to look at the graph of a function given by a formula, though. We can zoom in as close as we wish and still see a continuous curve or line. By using a high-power microscope, we can learn even more about rates and slopes.

High-power magnification is possible with a formula

Consider this rather complicated-looking function:

$$f(x) = \frac{2 + x^3 \cos x + 1.5^x}{2 + x^2}.$$

Let's find $f'(27)$, the rate of change of f when $x = 27$. We need to zoom in on the graph of f at the point $(27, f(27)) = (27, 69.859043)$. We do this in stages, producing a succession of windows that run clockwise from the upper left. Notice how the graph gets straighter with each magnification.



We will need a way to describe the part of the graph that we see in a window. Let's call it the **field of view**. The field of view of each window is only one-tenth as wide as the previous one, and the field of view of the last window is only one-millionth of the first! The last window shows what we would see if we looked at the original graph with a million-power microscope.

The field of view of a window

The microscope used to study functions is real, but it is different from the one a biologist uses to study micro-organisms. Our microscope is a computer graphing program that can "zoom in" on any point on a graph. The computer screen is the window you look through, and you determine the field of view when you set the size of the interval over which the graph is plotted.

Here is our point of departure: *the rate* $f'(27)$ *is the slope of the graph of* $f(x)$ *at* $x = 27$ *when we magnify the graph enough to make it look straight.* But how much is enough? Which window should we use? The following table gives the slope $\Delta y/\Delta x$ of the line that appears in each of the last four windows in the sequence. For Δx we take the difference between the x-coordinates of the points at the ends of the line, and for Δy we take the difference between the y-coordinates. In particular, the width of the field of view in each case is Δx.

Δx	Δy	$\Delta y/\Delta x$
.04	$-1.081\,508\,24 \times 10^{-2}$	$-.270\,377\,066$
.004	$-1.089\,338\,27 \times 10^{-3}$	$-.272\,334\,556$
.0004	$-1.089\,416\,49 \times 10^{-4}$	$-.272\,354\,131$
.00004	$-1.089\,417\,28 \times 10^{-5}$	$-.272\,354\,327$

As you can see, it *does* matter how much we magnify. The slopes $\Delta y/\Delta x$ in the table are not quite the same, so we don't yet have a definite value for $f'(27)$. The table gives us an idea how we *can* get a definite value, though. Notice that the slopes get more and more alike, the more we magnify. In fact, under successive magnifications the first five digits of $\Delta y/\Delta x$ have **stabilized**. We saw in chapter 2 how to think about a sequence of numbers whose digits stabilize one by one. We should treat the values of $\Delta y/\Delta x$ as **successive approximations** to the slope of the graph. The exact value of the slope is then the **limit** of these approximations as the width of the field of view shrinks to zero:

The slope is a limit

$$f'(27) = \text{the slope of the graph} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In the limit process we take $\Delta x \to 0$ because Δx is the width of the field of view. Since five digits of $\Delta y/\Delta x$ have stabilized, we can write

$$f'(27) = -.27235\ldots.$$

To find $f'(x)$ at some other point x, proceed the same way. Magnify the graph at that point repeatedly, until the value of the slope stabilizes. The method is very powerful. In the exercises you will have an opportunity to use it with other functions.

By using a microscope of arbitrarily high power, we have obtained further insights about rates and slopes. In fact, with these insights we can now state definitively what we mean by the slope of a curved graph and the rate of change of a function.

Definition. The **slope** of a graph at a point is the *limit* of the slopes seen in a microscope at that point, as the field of view shrinks to zero.

Definition. The **rate of change** of a function at a point is the slope of its graph at that point. Thus the rate of change is also a limit.

To calculate the value of the slope of the graph of $f(x)$ when $x = a$, we have to carry out a limit process. We can break down the process into these four steps:

1. Magnify the graph at the point $(a, f(a))$ until it appears straight.
2. Calculate the slope of the magnified segment.
3. Repeat steps 1 and 2 with successively higher magnifications.
4. Take the limit of the succession of slopes produced in step 3.

Local Linearity

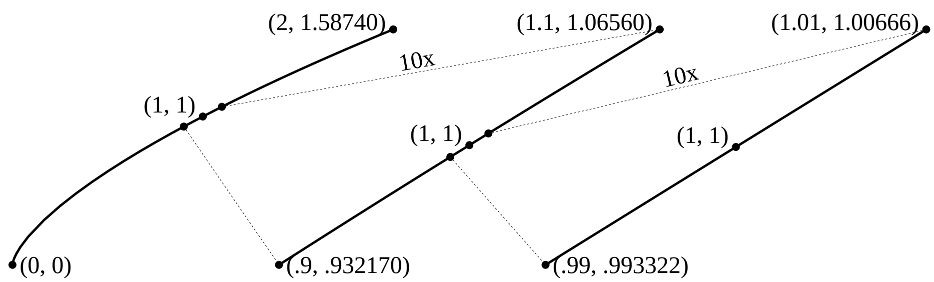
A microscope gives a local view

The crucial property of a microscope is that it allows us to look at a graph **locally**, that is, in a small neighborhood of a particular point. The two functions we have been studying in this section have curved graphs—like most functions. But *locally*, their graphs are straight—or nearly so. This is a remarkable property, and we give it a name. We say these functions are **locally linear**. In other words, a locally linear function looks like a linear function, locally.

The graph of a linear function has a definite slope at every point, and so does a *locally* linear function. For a linear function, the slope is easy to calculate, and it has the same value at every point. For a locally linear function, the slope is harder to calculate; it involves a limit process. The slope also varies from point to point.

All the standard functions are locally linear at almost all points

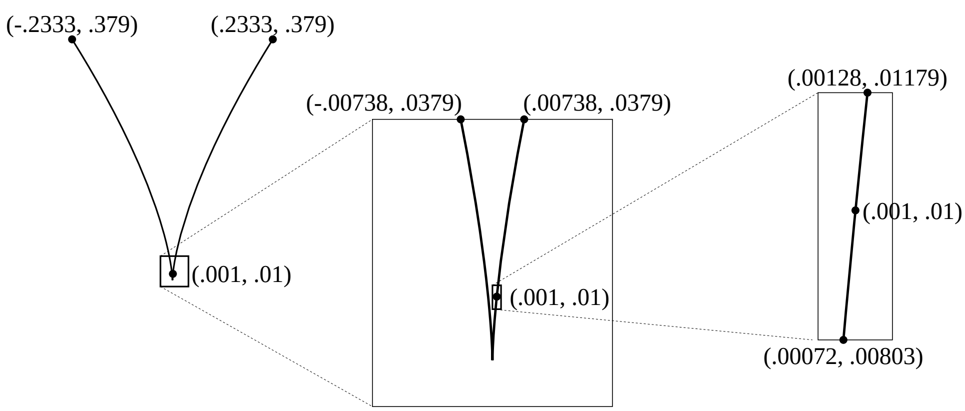
How common is local linearity? All the standard functions you already deal with are locally linear almost everywhere. To see why we use the qualifying phrase "almost everywhere," look at what happens when we view the graph of $y = f(x) = x^{2/3}$ with a microscope. At any point other than the origin, the graph is locally linear. For instance, if we view this graph over the interval from 0 to 2 and then zoom in on the point $(1, 1)$ by two successive powers of 10, here's what we see:



As the field of view shrinks, the graph looks more and more like a straight line. Using the highest magnification given, we estimate the slope of the graph—and hence the rate of change of the function—to be

$$f'(1) \approx \frac{\Delta y}{\Delta x} = \frac{1.006656 - .993322}{1.01 - .99} = \frac{.013334}{.02} = .6667.$$

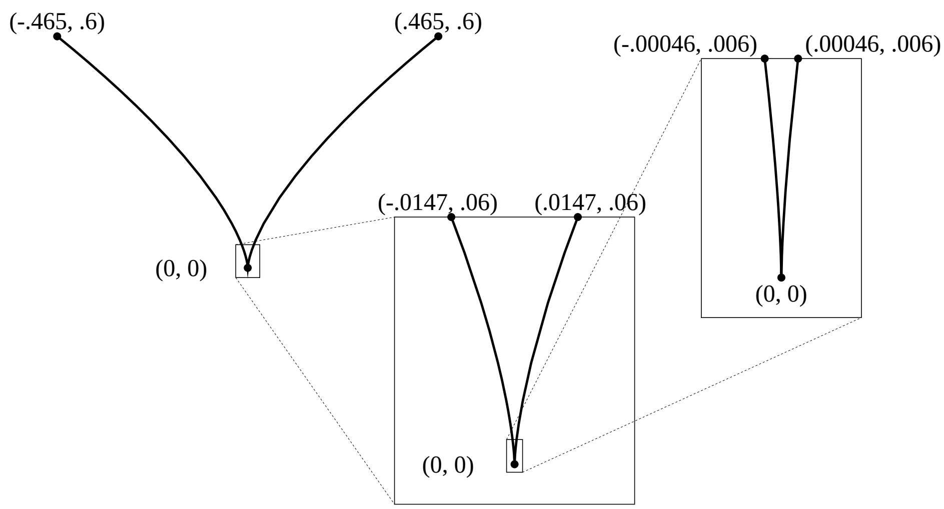
Similarly, if we zoom in on the point $(.001, .01)$ we get:



In the last window the graph looks like a line of slope

$$f'(.001) \approx \frac{.0118 - .0082}{.00128181 - .00074254} = \frac{.0036}{.00053937} = 6.674.$$

At the origin, though, something quite different happens:



The graph simply looks more and more sharply pointed the closer we zoom in to the origin—it never looks like a straight line. However, the origin turns out to be the only point where the graph does not eventually look like a straight line.

Fractals are locally *non*-linear objects

In spite of these examples, it is important to realize that local linearity is a very special property. There are some functions that fail to be locally linear anywhere! Such functions are called **fractals**. No matter how much you magnify the graph of a fractal at any point, it continues to look non-linear—bent and "pointy" in various ways. In recent years fractals have been used in problems where the more common (locally linear) functions are inadequate. For instance, they describe irregular shapes like coastlines and clouds, and they model the way molecules are knocked about in a fluid (this is called *Brownian motion*). However, calculus does not deal with such functions. On the contrary:

Calculus studies functions that are locally linear almost everywhere.

Exercises

Using a microscope

1. Use a computer microscope to do the following. (A suggestion: first look at each graph over a fairly large interval.)

a) With a window of size $\Delta x = .002$, estimate the rate $f'(1)$ where $f(x) = x^4 - 8x$.

b) With a window of size $\Delta x = .0002$, estimate the rate $g'(0)$ where $g(x) = 10^x$.

c) With a window of size $\Delta t = .05$, estimate the slope of the graph of $y = t + 2^{-t}$ at $t = 7$.

d) With a window of size $\Delta z = .0004$, estimate the slope of the graph of $w = \sin z$ at $z = 0$.

2. Use a computer microscope to determine the following values, correct to one decimal place. Obtain estimates using a *sequence* of windows, and shrink the field of view until the first two decimal places stabilize. Show *all* the estimates you constructed in each sequence.

a) $f'(1)$ where $f(x) = x^4 - 8x$.

b) $h'(0)$ where $h(s) = 3^s$.

c) The slope of the graph of $w = \sin z$ at $z = \pi/4$.

d) The slope of the graph of $y = t + 2^{-t}$ at $t = 7$.

e) The slope of the graph of $y = x^{2/3}$ at $x = -5$.

3. For each of the following functions, magnify its graph at the indicated point until the graph appears straight. Determine the equation of that straight line. Then verify that your equation is correct by plotting it as a second function in the same window you are viewing the given function. (The two graphs should "share phosphor"!)

a) $f(x) = \sin x$ at $x = 0$;

b) $\varphi(t) = t + 2^{-t}$ at $t = 7$;

c) $H(x) = x^{2/3}$ at $x = -5$.

4. Consider the function that we investigated in the text:

$$f(x) = \frac{2 + x^3 \cos x + 1.5^x}{2 + x^2}.$$

a) Determine $f(0)$.

b) Make a sketch of the graph of f on the interval $-1 \le x \le 1$. Use the same scale on the horizontal and vertical axes so your graph shows slopes accurately.

c) Sketch what happens when you magnify the previous graph so the field of view is only $-.001 \le x \le .001$.

d) Estimate the slope of the line you drew in the part (c).

e) Estimate $f'(0)$. How many decimal places of accuracy does your estimate have?

f) What is the equation of the line in part (c)?

5. A function that occurs in several different contexts in physical problems is

$$g(x) = \frac{\sin x}{x}.$$

Use a graphing program to answer the following questions.

a) Estimate the rate of change of g at the following points to two decimal place accuracy:

$$g'(1), \qquad g'(2.79), \qquad g'(\pi), \qquad g'(3.1).$$

b) Find three values of x where $g'(x) = 0$.

c) In the interval from 0 to 2π, where is g decreasing the most rapidly? At what rate is it decreasing there?

d) Find a value of x for which $g'(x) = -\,0.25$.

e) Although $g(0)$ is not defined, the function $g(x)$ seems to behave nicely in a neighborhood of 0. What seems to be true about $g(x)$ and $g'(x)$ when x is near 0?

f) According to your graphs, what value does $g(x)$ approach as $x \to 0$? What value does $g'(x)$ approach as $x \to 0$?

Rates from graphs; graphs from rates

6. a) Sketch the graph of a function f that has $f(1) = 1$ and $f'(1) = 2$.

b) Sketch the graph of a function f that has $f(1) = 1$, $f'(1) = 2$, and $f(1.1) = -5$.

7. A and B start off at the same time, run to a point 50 feet away, and return, all in 10 seconds. A graph of distance from the starting point as a function of time for each runner appears below. It tells where each runner is during this time interval.

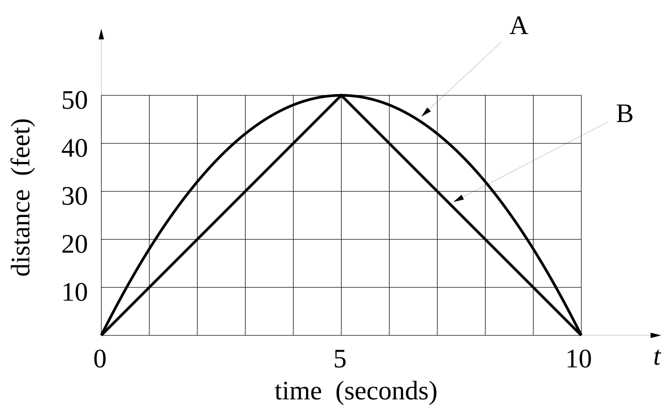
a) Who is in the lead during the race?

b) At what time(s) is A farthest ahead of B? At what time(s) is B farthest ahead of A?

c) Estimate how fast A and B are going after one second.

d) Estimate the velocities of A and B during each of the ten seconds. Be sure to assign *negative* velocities to times when the distance to the starting point is *shrinking*. Use these estimates to sketch graphs of the velocities of A and B versus time. (Although the velocity of B changes rapidly around $t = 5$, assume that the graph of B's distance *is* locally linear at $t = 5$.)

e) Use your graphs in (d) to answer the following questions. When is A going faster than B? When is B going faster than A? Around what time is A running at -5 feet/second (i.e., running 5 feet/second *toward* the starting point)?



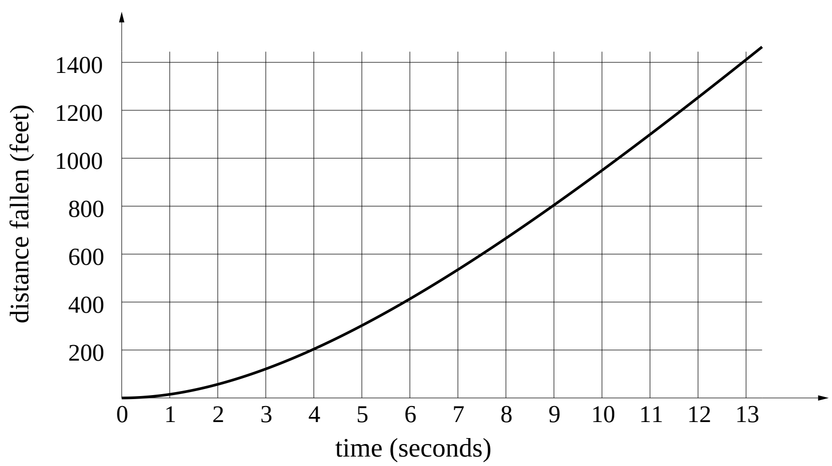
8. For each of the following functions draw a graph that reflects the given information. Restate the given information in the language and notation of rate of change, paying particular attention to the units in which any rate of change is expressed.

a) A woman's height h (in inches) depends on her age t (in years). Babies grow very rapidly for the first two years, then more slowly until the adolescent growth spurt; much later, many women actually become shorter because of loss of cartilage and bone mass in the spinal column.

b) The number R of rabbits in a meadow varies with time t (in years). In the early years food is abundant and the rabbit population grows rapidly. However, as the population of rabbits approaches the "carrying capacity" of the meadow environment, the growth rate slows, and the population never exceeds the carrying capacity. Each year, during the harsh conditions of winter, the population dies back slightly, although it never gets quite as low as its value the previous year.

c) In a fixed population of couples who use a contraceptive, the average number N of children per couple depends on the effectiveness E (in percent) of the contraceptive. If the couples are using a contraceptive of low effectiveness, a small increase in effectiveness has a small effect on the value of N. As we look at contraceptives of greater and greater effectiveness, small additional increases in effectiveness have larger and larger effects on N.

9. If we graph the distance travelled by a parachutist in freefall as a function of the length of time spent falling, we would get a picture something like the following:



a) Use this graph to make estimates of the parachutist's velocity at the end of each second.

b) Describe what happens to the velocity as time passes.

c) How far do you think the parachutist would have fallen by the end of 15 seconds?

10. True or false. If you think a statement is true, give your reason; if you think a statement is false, give a counterexample—i.e., an example that

shows why it must be false.

a) If $g'(t)$ is positive for all t, we can conclude that $g(214)$ is positive.

b) If $g'(t)$ is positive for all t, we can conclude that $g(214) > g(17)$.

c) Bill and Samantha are driving separate cars in the same direction along the same road. At the start Samantha is 1 mile in front of Bill. If their speeds are the same at every moment thereafter, at the end of 20 minutes Samantha will be 1 mile in front of Bill.

d) Bill and Samantha are driving separate cars in the same direction along the same road. They start from the same point at 10 am and arrive at the same destination at 2 pm the same afternoon. At some time during the four hours their speeds must have been exactly the same.

When local linearity fails

11. The absolute value function $f(x) = |x|$ is *not* locally linear at $x = 0$. Explore this fact by zooming in on the graph at $(0, 0)$. Describe what you see in successively smaller windows. Is there any change?

12. Find three points where the function $f(x) = |\cos x|$ fails to be locally linear. Sketch the graph of f to demonstrate what is happening.

13. Zoom in on the graph of $y = x^{4/5}$ at $(0, 0)$. In order to get an accurate picture, be sure that you use the same scales on the horizontal and vertical axes. Sketch what you see happening in successive windows. Is the function $x^{4/5}$ locally linear at $x = 0$?

14. Is the function $x^{4/5}$ locally linear at $x = 1$? Explain your answer.

15. This question concerns the function $K(x) = x^{10/9}$.

a) Sketch the graph of K on the interval $-1 \le x \le 1$. Compare K to the absolute value function $|x|$. Are they similar or dissimilar? In what ways? Would you say K is locally linear at the origin, or not?

b) Magnify the graph of K at the origin repeatedly, until the field of view is no bigger than $\Delta x \le 10^{-10}$. As you magnify, be sure the scales on the horizontal and vertical axes remain the same, so you get a true picture of the slopes. Sketch what you see in the final window.

c) After using the microscope do you change your opinion about the local linearity of K at the origin? Explain your response.

3.3 The Derivative

Definition

One of our main goals in this chapter is to make precise the notion of the rate of change of a function. In fact, we have already done that in the last section. We defined the rate of change of a function at a point to be the slope of its graph at that point; we defined the slope, in turn, by a four-step limit process. Thus, the precise definition of a rate of change involves a limit, and it involves geometric visualization—we think of a rate as a slope. We introduce a new word—*derivative*—to embrace both of these concepts as we now understand them.

> **Definition**. The **derivative** of the function $f(x)$ at $x = a$ is its rate of change at $x = a$, which is the same as the slope of its graph at $(a, f(a))$. The derivative of f at a is denoted $\boldsymbol{f'(a)}$.

Later in this section we will extend our interpretation of the derivative to include the idea of a *multiplier*, as well as a rate and a slope. Besides providing us with a single word to describe rates, slopes, and multipliers, the term "derivative" also reminds us that the quantity $f'(a)$ is *derived* from information about the function f in a particular way. It is worth repeating here the four steps by which we derive $f'(a)$:

1. Magnify the graph at the point $(a, f(a))$ until it appears straight.
2. Calculate the slope of the magnified segment.
3. Repeat steps 1 and 2 with successively higher magnifications.
4. Take the limit of the succession of slopes produced in step 3.

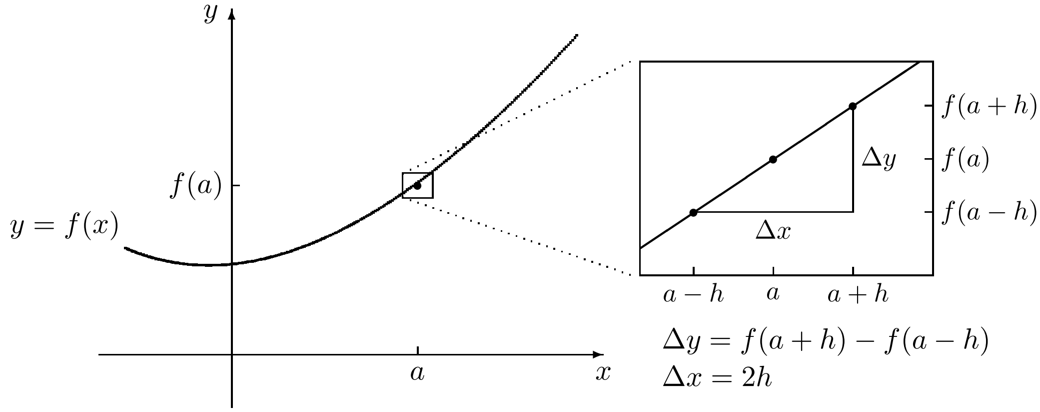
We can express this limit in analytic form in the following way:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

The **difference quotient**

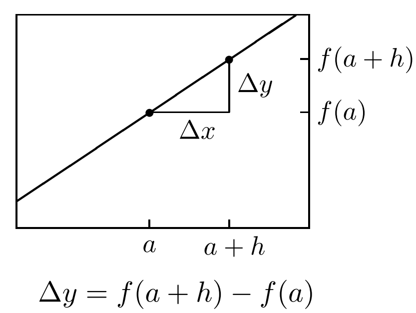
$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a-h)}{2h}$$

is the usual way we estimate the slope of the magnified graph of f at the point $(a, f(a))$. As the following figure shows, the calculation involves two points equally spaced on either side of $(a, f(a))$.



Estimating the value of the derivative $f'(a)$

Choosing points in a window. To estimate the slope in the window above, we chose two particular points, $(a-h, f(a-h))$ and $(a+h, f(a+h))$. However, *any* two points in the window would give us a valid estimate. Our choice depends on the situation. For example, if we are working with formulas, we want simple expressions. In that case we would probably replace $(a-h, f(a-h))$ by $(a, f(a))$. We do that in the window on the left. The resulting slope is



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

While the limiting value of $\Delta y/\Delta x$ doesn't depend on the choices you make, the *estimates* you produce with a fixed Δx can be closer to or farther from the true value. The exercises will explore this.

Data versus formulas. The derivative is a limit. To find that limit we have to be able to zoom in arbitrarily close, to make Δx arbitrarily small. For functions given by data, that is usually impossible; we can't use any Δx smaller than the spacing between the data points. Thus, a data function of this sort does not have a derivative, strictly speaking. However, by zooming in as much as the data allow, we get the most precise description possible for the rate of change of the function. In these circumstances it makes a difference which points we choose in a window to calculate $\Delta y/\Delta x$. In the exercises you will have a chance to see how the precision of your estimate depends on which points you choose to calculate the slope.

A data function might not have a derivative

There are rules for finding derivatives

For a function given by a formula, it is possible to find the value of the derivative exactly. In fact, the derivative of a function given by a formula is itself given by a formula. Later in this chapter we will describe some general rules which will allow us to produce the formula for the derivative without going through the successive approximation process each time. In chapter 5 we will discuss these rules more fully.

Practical considerations. The derivative is a limit, and there are always practical considerations to raise when we discuss limits. As we saw in chapter 2, we cannot expect to construct the entire decimal expansion of a limit. In most cases all we can get are a specified number of digits. For example, in section 2 we found that

$$\text{if } f(x) = \frac{2 + x^3 \cos x + 1.5^x}{2 + x^2}, \text{ then } f'(27) = -.27235\ldots.$$

The same digits without the "..." give us **approximations**. Thus we can write $f'(27) \approx -.27235$; we also have $f'(27) \approx -.2723$ and $f'(27) \approx -.272$. Which approximation is the right one to use depends on the context. For example, if f appears in a problem in which all the other quantities are known only to one or two decimal places, we probably don't need a very precise value for $f'(27)$. In that case we don't have to carry the sequence of slopes $\Delta y/\Delta x$ very far. For instance, if we want to know $f'(27)$ to three decimal places, and so justify writing $f'(27) \approx -.272$, we only need to continue the zooming process until the slopes $\Delta y/\Delta x$ all have values that begin $-.2723\ldots$. By the table on page 111, $\Delta x = .0004$ is sufficient.

Language and Notation

• If f has a derivative at a, we also say ***f* is differentiable at *a***. If f is differentiable at every point a in its domain, we say ***f* is differentiable**.
• Do *locally linear* and *differentiable* mean the same thing? The awkward case is a function whose graph is vertical at a point (for example, $y = \sqrt[3]{x}$ at the origin). On the one hand, it makes sense to say that the function is locally linear at such a point, because the graph looks straight under a microscope. On the other hand, the derivative itself is undefined, because the line is vertical. So the function is locally linear, but not differentiable, at that point.

There is another way to view the matter. We can say, instead, that a vertical line *does* have a slope, and its value is *infinity* (∞). From this point of view, if the graph of f is vertical at $x = a$, then $f'(a) = \infty$. In other words, f *does* have a derivative at $x = a$; its value just happens to be ∞.

Which view is "right"? Neither; we can choose either. Our choice is a matter of convention. (In some countries cars travel on the left; in others, on the right. That's a convention, too.) However, we will follow the second alternative. One advantage is that we will be able to use the derivative to indicate where the graph of a function is vertical. Another is that *locally linear* and *differentiable* then mean exactly the same thing.

Convention: if the graph of f is vertical at a, write $f'(a) = \infty$

• Suppose $y = f(x)$ and the quantities x and y appear in a context in which they have units. Then the derivative of $f'(x)$ *also* has units, because it is the rate of change of y with respect to x. The units for the derivative must be

$$\text{units for } f' = \frac{\text{units for output } y}{\text{units for input } x}.$$

We have already seen several examples—persons per day, miles per hour, milligrams per pound, dollars of profit per dollar change in price—and we will see many more.

• There are several notations for the derivative. You should be aware of them because they are all in common use and because they reflect different ways of viewing the derivative. We have been writing the derivative of $y = f(x)$ as $f'(x)$. Leibniz wrote it as dy/dx. This notation has several advantages. It resembles the quotient $\Delta y/\Delta x$ that we use to approximate the derivative. Also, because dy/dx looks like a rate, it helps remind us that a derivative is a rate. Later on, when we consider the chain rule to find derivatives, you'll see that it can be stated very vividly using Leibniz's notation.

Leibniz's notation

The German philosopher Gottfried Wilhelm Leibniz (1646-1716) developed calculus about the same time Newton did. While Newton dealt with derivatives in more or less the way we do, Leibniz introduced a related idea which he called a *differential*—'infinitesimally small' numbers which he would write as dx and dy.

The other notation still encountered is due to Newton. It occurs primarily in physics and is used to denote rates with respect to time. If a quantity y is changing over time, then the Newton notation expresses the derivative of y as $\dot{y}$ (that's the variable y with a dot over it).

Newton's notation

The Microscope Equation

A Context: Driving Time

If you make a 400 mile trip at an average speed of 50 miles per hour, then the trip takes 8 hours. Suppose you increase the average speed by 2 miles per hour. How much time does that cut off the trip?

One way to approach this question is to start with the basic formula

$$\text{speed} \times \text{time} = \text{distance}.$$

Travel time depends on speed

The distance is known to be 400 miles, and we really want to understand how *time* T depends on *speed* s. We get T as a function of s by rewriting the last equation:

$$T \text{ hours} = \frac{400 \text{ miles}}{s \text{ miles per hour}}.$$

To answer the question, just set $s = 52$ miles per hour in this equation. Then $T = 7.6923$ hours, or about 7 hours, 42 minutes. Thus, compared to the original 8 hours, the higher speed cuts 18 minutes off your driving time.

What happens to the driving time if you increase your speed by 4 miles per hour, or 5, instead of 2? What happens if you go slower, say 2 or 3 miles per hour slower? We could make a fresh start with each of these questions and answer them, one by one, the same way we did the first. But taking the questions one at a time misses the point. What we really want to know is the general pattern:

How does travel time respond to changes in speed?

If I'm travelling at 50 miles per hour, how much does any given increase in speed decrease my travel time?

We already know how T and s are related: $T = 400/s$ hours. This question, however, is about the connection between a *change* in speed of

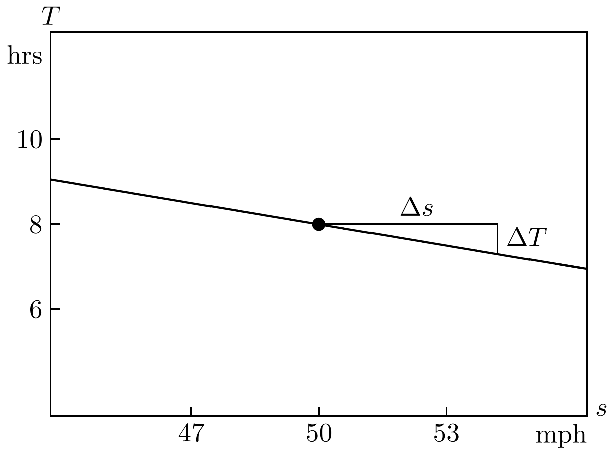
$$\Delta s = s - 50 \text{ miles per hour}$$

and a *change* in arrival time of

$$\Delta T = T - 8 \text{ hours}.$$

To answer it we should change our point of view slightly. It is not the relation between s and T, but between Δs and ΔT, that we want to understand.

Since we are considering speeds s that are only slightly above or below 50 miles per hour, Δs will be small. Consequently, the arrival time T will be only slightly different from 8 hours, so ΔT will also be small. Thus we want to study small changes in the function $T = 400/s$ near $(s, T) = (50, 8)$. The natural tool to use is a microscope.



How travel time changes with speed around 50 miles per hour

In the microscope window above we see the graph of $T = 400/s$, magnified at the point $(s, T) = (50, 8)$. The field of view was chosen so that s can take values about 6 mph above or below 50 mph. The graph looks straight, and its slope is $T'(50)$. In the exercises you are asked to determine the value of $T'(50)$; you should find $T'(50) \approx -.16$. (Later on, when we have rules for finding the derivative of a formula, you will see that $T'(50) = -.16$ *exactly*.) Since the quotient $\Delta T/\Delta s$ is also an estimate for the slope of the line in the window, we can write

The slope of the graph in the microscope window

$$\frac{\Delta T}{\Delta s} \approx -.16 \text{ hours } \textit{per} \text{ mile per hour.}$$

If we multiply both sides of this approximate equation by Δs miles per hours, we get

$$\Delta T \approx -.16\, \Delta s \text{ hours.}$$

This equation answers our question about the general pattern relating changes in travel time to changes in speed. It says that the changes are *proportional*. For every mile per hour increase in speed, travel time decreases by about .16 hours, or about $9\frac{1}{2}$ minutes. Thus, if the speed is 1 mph over

How travel time changes with speed

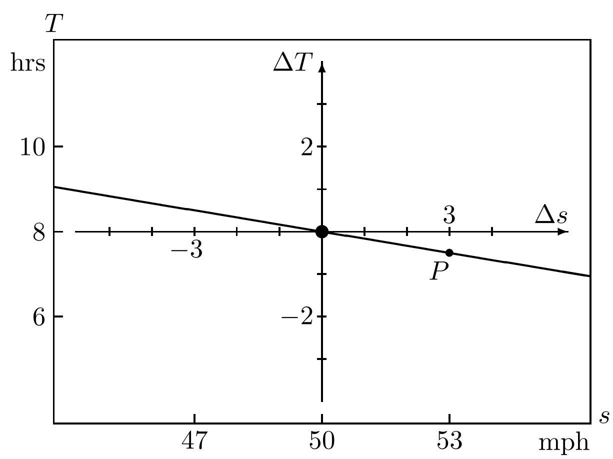
50 mph, travel time is cut by about $9\frac{1}{2}$ minutes. If we double the increase in speed, that doubles the savings in time: if the speed is 2 mph over 50 mph, travel time is cut by about 19 minutes. Compare this with a value of about 18 minutes that we got with the exact equation $T = 400/s$.

Δs and ΔT now have a special meaning

Notice that we are using Δs and ΔT in a slightly more restricted way than we have previously. Up to now, Δs measured the horizontal distance between *any* two points on a graph. Now, however, Δs just measures the horizontal distance from the fixed point $(s, T) = (50, 8)$ (marked with a large dot) that sits at the center of the window. Likewise, ΔT just measures the vertical distance from this point. The central point therefore plays the role of an origin, and Δs and ΔT are the *coordinates* of a point measured from that origin. To underscore the fact that Δs and ΔT are really coordinates, we have added a Δs-axis and a ΔT-axis in the window below. Notice that these coordinate axes have their own labels and scales.

Δs and ΔT are coordinates in the window

Every point in the window can therefore be described in two different coordinate systems. The two different sets of coordinates of the point labelled P, for instance, are $(s, T) = (53, 7.52)$ and $(\Delta s, \Delta T) = (3, -.48)$. The first pair says "When your speed is 53 mph, the trip will take 7.52 hrs." The second pair says "When you increase your speed by 3 mph, you will decrease travel time by .48 hrs." Each statement can be translated into the other, but each statement has its own point of reference.



The microscope equation: $\Delta T \approx -.16\,\Delta s$ hours

We call $\Delta T \approx -.16\,\Delta s$ the **microscope equation** because it tells us how the microscope coordinates Δs and ΔT are related.

In fact, we now have two different ways to describe how travel time is related to speed. They can be compared in the following table.

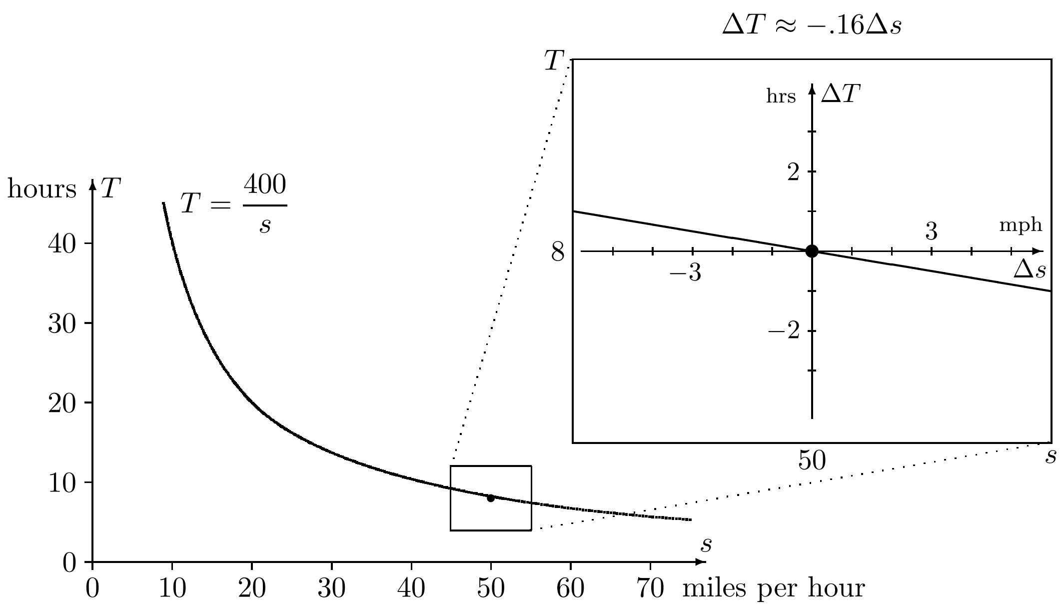
	GLOBAL	LOCAL
coordinates:	$s,\ T$	$\Delta s,\ \Delta T$
equation:	$T = 400/s$	$\Delta T \approx -.16\,\Delta s$
properties:	exact non-linear	approximate linear

Global vs. local descriptions

We say the microscope equation is *local* because it is intended to deal only with speeds near 50 miles per hour. There is a different microscope equation for speeds near 40 miles per hour, for instance. By contrast, the original equation is *global*, because it works for all speeds. While the global equation is exact it is also non-linear; this can make it more difficult to compute. The microscope equation is approximate but linear; it is easy to compute. It is also easy to put into words:

At 50 miles per hour, the travel time of a 400 mile journey decreases $9\frac{1}{2}$ minutes for each mile per hour increase in speed.

The connection between the global equation and the microscope equation is shown in the following illustration.



Local Linearity and Multipliers

The reasoning that led us to a microscope equation for travel time can be applied to any locally linear function. If $y = f(x)$ is locally linear, then at $x = a$ we can write

$$\textbf{The microscope equation:} \quad \Delta y \approx f'(a) \cdot \Delta x$$

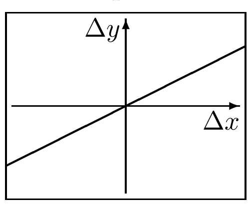
We know an equation of the form $\Delta y = m \cdot \Delta x$ tells us that y is a linear function of x, in which m plays the role of slope, rate, and multiplier. The microscope equation therefore tells us that **y is a linear function of x when x is near a**—at least approximately. In this almost-linear relation, the derivative $f'(a)$ plays the role of slope, rate, and multiplier.

The microscope equation is the analytic form of local linearity

The microscope equation is just the idea of local linearity expressed analytically rather than geometrically—that is, by a formula rather than by a picture. Here is a chart that shows how the two descriptions of local linearity fit together.

$y = f(x)$ is locally linear at $x = a$:

GEOMETRICALLY	ANALYTICALLY
When magnified at $(a, f(a))$, the graph of f is almost straight, and the slope of the line is $f'(a)$.	When x is near a, y is almost a linear function of x, and the multiplier is $f'(a)$.
microscope window	microscope equation

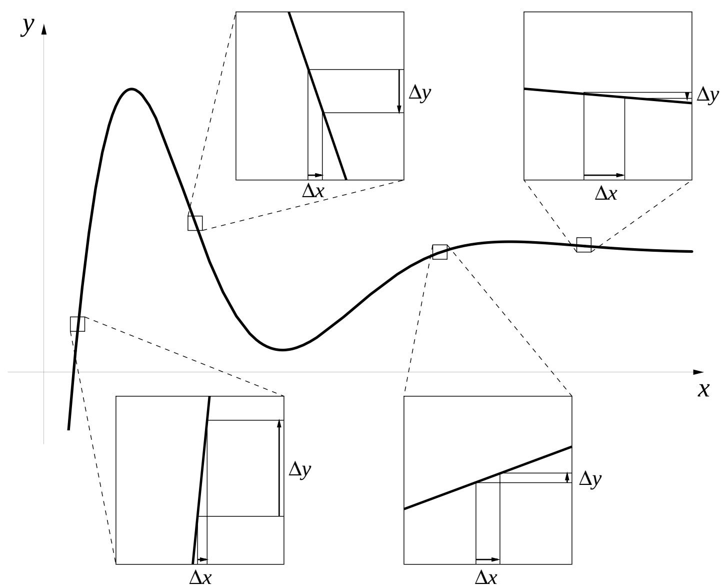


$$\Delta y \approx f'(a) \cdot \Delta x$$

Of course, the graph in a microscope window is not *quite* straight. The analytic counterpart of this statement is that the microscope equation is not *quite* exact—the two sides of the equation are only approximately equal. We write "≈" instead of "=". However, we can make the graph look even straighter by increasing the magnification—or, what is the same thing, by decreasing the field of view. Analytically, this increases the exactness of the microscope equation. Like a laboratory microscope, our microscope is most accurate at the center of the field of view, with increasing aberration toward the periphery!

In the microscope equation $\Delta y \approx f'(a) \cdot \Delta x$, **the derivative is the multiplier that tells how y responds to changes in x**. In particular, a small increase in x produces a change in y that depends on the sign and magnitude of $f'(a)$ in the following way:

- $f'(a)$ is large and positive $\Rightarrow$ large increase in y,
- $f'(a)$ is small and positive $\Rightarrow$ small increase in y,
- $f'(a)$ is large and negative $\Rightarrow$ large decrease in y,
- $f'(a)$ is small and negative $\Rightarrow$ small decrease in y.



For example, suppose we are told the value of the derivative is 2. Then any small change in x induces a change in y approximately twice as large. If, instead, the derivative is $-1/5$, then a small change in x produces a change in y only one fifth as large, and in the opposite direction. That is, if x increases, then y decreases, and vice-versa.

The microscope equation should look familiar to you. It has been with us from the beginning of the course. Our "recipe" $\Delta S = S' \cdot \Delta t$ for predicting future values of S in the S-I-R model is just the microscope equation for the function $S(t)$. (Although we wrote it with an "=" instead of an "≈" in the

The microscope equation is the recipe for building solutions to rate equations first chapter, we noted that ΔS would provide us only an *estimate* for the new value of S.) The success of Euler's method in producing solutions to rate equations depends fundamentally on the fact that the functions we are trying to find are locally linear.

The derivative is one of the fundamental concepts of the calculus, and one of its most important roles is in the microscope equation. Besides giving us a tool for building solutions to rate equations, the microscope equation helps us do estimation and error analysis, the subject of the next section.

We conclude with a summary that compares linear and locally linear functions. Note there are two differences, but only two: 1) the equation for local linearity is only an approximation; 2) it holds only locally—i.e., near a given point.

If $y = f(x)$ is linear, then $\Delta y = m \cdot \Delta x$; the constant m is rate, slope, and multiplier for all x.	**If $y = f(x)$ is locally linear, then $\Delta y \approx f'(a) \cdot \Delta x$; the derivative $f'(a)$ is rate, slope, and multiplier for x near a.**

Exercises

Computing Derivatives

1. Sketch graphs of the following functions and use these graphs to determine which function has a derivative that is always positive (except at $x = 0$, where neither the function nor its derivative is defined).

$$y = \frac{1}{x} \quad y = \frac{-1}{x} \quad y = \frac{1}{x^2} \quad y = \frac{-1}{x^2}$$

What feature of the graph told you whether the derivative was positive?

2. For each of the functions f below, approximate its derivative at the given value $x = a$ in two different ways. First, use a computer microscope (i.e., a graphing program) to view the graph of f near $x = a$. Zoom in until the graph looks straight and find its slope. Second, use a calculator to find the value of the quotient

$$\frac{f(a+h) - f(a-h)}{2h}$$

for $h = .1, .01, .001, \ldots, .000001$. Based on these values of the quotients, give your best estimate for $f'(a)$, and say how many decimal places of accuracy it has.

a) $f(x) = 1/x$ at $x = 2$.

b) $f(x) = \sin(7x)$ at $x = 3$.

c) $f(x) = x^3$ at $x = 200$.

d) $f(x) = 2^x$ at $x = 5$.

3. In a later section we will establish that the derivative of $f(x) = x^3$ at $x = 1$ is exactly 3: $f'(1) = 3$. This question concerns the freedom we have to choose points in a window to estimate $f'(1)$ (see page 121). Its purpose is to compare two quotients, to see which gets closer to the exact value of $f'(1)$ for a fixed "field of view" Δx. The two quotients are

$$Q_1 = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a-h)}{2h} \quad \text{and} \quad Q_2 = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

In this problem $a = 1$.

a) Construct a table that shows the values of Q_1 and Q_2 for each $h = 1/2^k$, where $k = 0, 1, 2, \ldots, 8$. If you wish, you can use this program to compute the values:

```
a = 1
FOR k = 0 TO 8
        h = 1 / 2 ^ k
        q1 = ((a + h) ^ 3 - (a - h) ^ 3) / (2 * h)
        q2 = ((a + h) ^ 3 - a ^ 3) / h
        PRINT h, q1, q2
NEXT k
```

b) How many digits of Q_1 stabilize in this table? How many digits of Q_2?

c) Which is a better estimator—Q_1 or Q_2? To indicate how much better, give the value of h for which the *better* estimator provides an estimate that is as close as the best estimate provided by the *poorer* estimator.

4. Repeat all the steps of the last question for the function $f(x) = \sqrt{x}$ at $x = 9$. The exact value of $f'(9)$ is $1/6$.

Comment: Note that, in section 1, we estimated the rate of change of the sunrise function using an expression like Q_1 rather than one like Q_2. The previous exercises should persuade you this was deliberate. We were trying to get the most representative estimates, given the fact that we could not reduce the size of Δx arbitrarily.

5. At this point you will find it convenient to write a more general derivative-finding program. You can modify the program in problem 3. to do this by having a `DEF` command at the beginning of the program to specify the function you are currently interested in. For instance, if you insert the command `DEF fnf (x) = x ^ 3` at the beginning of the program, how could you simplify the lines specifying `q1` and `q2`? If you then wanted to calculate the derivative of another function at a different x-value, you would only need to change the `DEF` specification and, depending on the point you were interested in, the `a = 1` line.

6. Use one of the methods of problem 2 to estimate the value of the derivative of each of the following functions at $x = 0$:

$$y = 2^x, \quad y = 3^x, \quad y = 10^x, \quad \text{and} \quad y = (1/2)^x.$$

These are called **exponential** functions, because the input variable x appears in the exponent. How many decimal places accuracy do your approximations to the derivatives have?

7. In this problem we look again at the exponential function $f(x) = 2^x$ from the previous problem.

a) Use the rules for exponents to put the quotient

$$\frac{f(a+h) - f(a)}{h}$$

in the simplest form you can.

b) We know that

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$

Use this fact, along with the algebraic result of part (a), to explain why $f'(a) = f'(0) \cdot 2^a$.

8. Apply all the steps of the previous question to the exponential function $f(x) = b^x$ with an arbitrary base b. Show that $f'(x) = f'(0) \cdot b^x$.

9. a) For which values of x is the absolute value function $y = |x|$ differentiable?

b) At each point where $y = |x|$ is differentiable, find the value of the derivative.

The microscope equation

10. Write the microscope equation for each of the following functions at the indicated point. (To find the necessary derivative, consult problem 2.)

a) $f(x) = 1/x$ at $x = 2$.

b) $f(x) = \sin(7x)$ at $x = 3$.

c) $f(x) = x^3$ at $x = 200$.

d) $f(x) = 2^x$ at $x = 5$.

11. This question uses the microscope equation for $f(x) = 1/x$ at $x = 2$ that you constructed in the previous question.

a) Draw the graph of what you would see in the microscope if the field of view is .2 units wide.

b) If we take $x = 2.05$, what is Δx in the microscope equation? What estimate does the microscope equation give for Δy? What estimate does the microscope equation then give for $f(2.05) = 1/2.05$? Calculate the true value of $1/2.05$ and compare the two values; how far is the microscope estimate from the true value?

c) What estimate does the microscope equation give for $1/2.02$? How far is this from the true value?

d) What estimate does the microscope equation give for $1/1.995$? How far is this from the true value?

12. This question concerns the travel time function $T = 400/s$ hours, discussed in the text.

a) How many hours does a 400-mile trip take at an average speed of 40 miles per hour?

b) Find the microscope equation for T when $s = 40$ miles per hour.

c) At what rate does the travel time decrease as speed increases around 40 mph—in hours *per* mile per hour?

d) According to the microscope equation, how much travel time is saved by increasing the speed from 40 to 45 mph?

e) According to the microscope equation based at 50 mph (as done in the text), how much time is *lost* by decreasing the speed from 50 to 45 mph?

f) The last two parts both predict the travel time when the speed is 45 mph. Do they give the same result?

13. a) Suppose $y = f(x)$ is a function for which $f(5) = 12$ and $f'(5) = .4$. Write the microscope equation for f at $x = 5$.

b) Draw the graph of what you would see in the microscope. Do you need a formula for f itself, in order to do this?

c) If $x = 5.3$, what is Δx in the microscope equation? What estimate does the microscope equation give for Δy? What estimate does the microscope equation then give for $f(5.3)$?

d) What estimates does the microscope equation give for the following: $f(5.23)$, $f(4.9)$, $f(4.82)$, $f(9)$? Do you consider these estimates to be equally reliable?

14. a) Suppose $z = g(t)$ is a function for which $g(-4) = 7$ and $g'(-4) = 3.5$. Write the microscope equation for g at $t = -4$.

b) Draw the graph of what you see in the microscope.

c) Estimate $g(-4.2)$ and $g(-3.75)$.

d) For what value of t near -4 would you estimate that $g(t) = 6$? For what value of t would you estimate $g(t) = 8.5$?

15. If $f(a) = b$, $f'(a) = -3$ and if k is small, which of the following is the best estimate for $f(a + k)$?

$$a + 3k,\ b + 3k,\ a + 3b,\ b - 3k,\ a - 3k,\ 3a - b,\ a^2 - 3b,\ f'(a + k)$$

16. If f is differentiable at a, which of the following, for small values of h, are reasonable estimates of $f'(a)$?

$$\frac{f(a+h) - f(a-h)}{h} \qquad \frac{f(a+h) - f(a-h)}{2h}$$
$$\frac{f(a+h) - f(h)}{h} \qquad \frac{f(a+2h) - f(a-h)}{3h}$$

17. Suppose a person has travelled D feet in t seconds. Then $D'(t)$ is the person's velocity at time t; $D'(t)$ has units of feet per second.

a) Suppose $D(5) = 30$ feet and $D'(5) = 5$ feet/second. Estimate the following:

$$D(5.1) \qquad D(5.8) \qquad D(4.7)$$

b) If $D(2.8) = 22$ feet, while $D(3.1) = 26$ feet, what would you estimate $D'(3)$ to be?

18. Fill in the blanks.

a) If $f(3) = 2$ and $f'(3) = 4$, a reasonable estimate of $f(3.2)$ is ____.

b) If $g(7) = 6$ and $g'(7) = .3$, a reasonable estimate of $g(6.6)$ is ____.

c) If $h(1.6) = 1$, $h'(1.6) = -5$, a reasonable estimate of $h($____$)$ is 0.

d) If $F(2) = 0$, $F'(2) = .4$, a reasonable estimate of $F($____$)$ is .15.

e) If $G(0) = 2$ and $G'(0) =$ ____, a reasonable estimate of $G(.4)$ is 1.6.

f) If $H(3) = -3$ and $H'(3) =$ ____, a reasonable estimate of $H(2.9)$ is -1.

19. In manufacturing processes the profit is usually a function of the number of units being produced, among other things. Suppose we are studying some small industrial company that produces n units in a week and makes a corresponding weekly profit of P. Assume $P = P(n)$.

a) If $P(1000) = \$500$ and $P'(1000) = \$2$/unit, then

$$P(1002) \approx \_\_\_\_\_\_ \qquad P(995) \approx \_\_\_\_\_\_ \qquad P(\_\_\_\_\_\_) \approx \$512$$

b) If $P(2000) = \$3000$ and $P'(2000) = -\$5$/unit, then

$$P(2010) \approx \_\_\_\_\_\_ \qquad P(1992) \approx \_\_\_\_\_\_ \qquad P(\_\_\_\_\_\_) \approx \$3100$$

c) If $P(1234) = \$625$ and $P(1238) = \$634$, then what is an estimate for $P'(1236)$?

3.4 Estimation and Error Analysis

Making Estimates

The Expanding House

In the book *The Secret House – 24 hours in the strange and unexpected world in which we spend our nights and days* (Simon and Schuster, 1986), David Bodanis describes many remarkable events that occur at the microscopic level in an ordinary house. At one point he explains how sunlight heats up the structure, stretching it imperceptibly in every direction through the day until it has become several cubic inches larger than it was the night before. Is it plausible that a house can become several cubic inches larger as it expands in the heat of the day? In particular, how much longer, wider, and taller would it have to become if it were to grow in volume by, let us say, 3 cubic inches?

How much does a house expand in the heat?

For simplicity, assume the house is a cube 200 inches on a side. (This is about 17 feet, so the house is the size of a small, two-story cottage.) If s is the length of a side of *any* cube, in inches, then its volume is

$$V = s^3 \text{ cubic inches.}$$

Our question is about how V changes with s when s is about 200 inches. In particular, we want to know which Δs would yield a ΔV of 3 cubic inches. This is a natural question for the microscope equation

$$\Delta V \approx V'(200) \cdot \Delta s.$$

According to exercise 2c in the previous section, we can estimate the value of $V'(200)$ to be about 120,000, and the appropriate units for V' are cubic inches per inch. Thus

$$\begin{aligned} \Delta V &\approx 120000\,\Delta s \\ 3 \text{ cubic inches} &\approx 120000\,\frac{\text{cubic inches}}{\text{inches}} \times \Delta s \text{ inches,} \end{aligned}$$

so $\Delta s \approx 3/120000 = .000025$ inches—many times thinner than a human hair!

This value is much too small. To get a more realistic value, let's suppose the house is made of wood and the temperature increases about 30°F from night to day. Then measurements show that a 200-inch length of wood will actually become about $\Delta s = .01$ inches longer. Consequently the volume will actually expand by about

$$\Delta V \approx 120000 \frac{\text{cubic inches}}{\text{inches}} \times .01 \text{ inches} = 1200 \text{ cubic inches.}$$

This increase is 400 times as much as Bodanis claimed; it is about the size of a small computer monitor. So even as he opens our eyes to the effects of thermal expansion, Bodanis dramatically understates his point.

Estimates versus Exact Values

Don't lose sight of the fact that the values we derived for the expanding house are *estimates*. In some cases we can get the exact values. Why don't we, whenever we can?

For example, we can calculate exactly how much the volume increases when we add $\Delta s = .01$ inches to $s = 200$ inches. The increase is from $V = 200^3 = 8{,}000{,}000$ cubic inches to

$$V = (200.01)^3 = 8001200.060001 \text{ cubic inches.}$$

Thus, the *exact* value of ΔV is 1200.060001 cubic inches. The estimate is off by only about .06 cubic inches. This isn't very much, and it is even less significant when you think of it as a percentage of the volume (namely 1200 cubic inches) being calculated. The percentage is

$$\frac{.06 \text{ cubic inches}}{1200 \text{ cubic inches}} = .00005 = .005\%.$$

That is, the difference is only 1/200 of 1% of the calculated volume.

To get the exact value we had to cube two numbers and take their difference. To get the estimate we only had to do a single multiplication. Estimates made with the microscope equation are always easy to calculate—they involve only linear functions. Exact values are usually harder to calculate. As you can see in the example, the extra effort may not gain us extra information. That's one reason why we don't always calculate exact values when we can.

Exact values can be harder to calculate than estimates.

Here's another reason. Go back to the question: How large must Δs be if $\Delta V = 3$ cubic inches? To get the exact answer, we must solve for Δs in the equation

$$\begin{aligned} 3 = \Delta V &= (200 + \Delta s)^3 - 200^3 \\ &= 200^3 + 3(200)^2 \Delta s + 3(200)(\Delta s)^2 + (\Delta s)^3 - 200^3. \end{aligned}$$

Simplifying, we get

$$3 = 120000\,\Delta s + 600(\Delta s)^2 + (\Delta s)^3.$$

This is a cubic equation for Δs; it *can* be solved, but the steps are complicated. Compare this with solving the microscope equation:

$$3 = 120000\,\Delta s.$$

Thus, another reason we don't calculate exact values at every opportunity is that the calculations can be daunting. The microscope estimates are always straightforward.

Perhaps the most important reason, though, is the insight that calculating V' gave us. Let's translate into English what we have really been talking about:

> In dealing with a cube 200 inches on a side, any small change (measured in inches) in the length of the sides produces a change (measured in cubic inches) in the volume approximately 120,000 times as great.

This is, of course, simply another instance of the point we have made before, that small changes in the input and the output are related in an (almost) linear way, even when the underlying function is complex. Let's continue this useful perspective by looking at error analysis.

Propagation of Error

From Measurements to Calculations

We can view all the estimates we made for the expanding house from another perspective—the lack of precision in measurement. To begin with, just think of the house as a cubical box that measures 200 inches on a side. Then the volume must be 8,000,000 cubic inches. But measurements are never exact, and any uncertainty in measuring the length of the side will lead to an uncertainty in calculating the volume. Let's say your measurement of length is accurate to within .5 inch. In other words, you believe the true length lies between 199.5 inches and 200.5 inches, but you are uncertain precisely where it lies within that interval. How uncertain does that make your calculation of the volume?

How uncertain is the calculated value of the volume?

There is a direct approach to this question: we can simply say that the volume must lie between 199.5^3 = 7,940,149.875 cubic inches and 200.5^3 = 8,060,150.125 cubic inches. In a sense, these values are almost too precise. They don't reveal a general pattern. We would like to know how an uncertainty—or error—in measuring the length of the side of a cube propagates to an error in calculating its volume.

Let's take another approach. If we measure s as 200 inches, and the true value differs from this by Δs inches, then Δs is the error in measurement. That produces an error ΔV in the calculated value of V. The microscope equation for the expanding house (page 136) tells us how ΔV depends on Δs when $s = 200$:

$$\Delta V \approx 120000\, \Delta s.$$

Since we now interpret Δs and ΔV as errors, the microscope equation becomes the **error propagation** equation:

$$\text{error in } V \text{ (cu. in.)} \approx 120000 \left(\frac{\text{cu. in.}}{\text{inch}}\right) \times \text{error in } s \text{ (inches)}.$$

The microscope equation describes how errors propagate

Thus, for example, an error of 1/2 inch in measuring s propagates to an error of about 60,000 cubic inches in calculating V. This is about 35 cubic feet, the size of a large refrigerator! Putting it another way:

$$\text{if } s = 200 \pm 0.5 \text{ inches, then } V \approx 8,000,000 \pm 60,000 \text{ cubic inches.}$$

If we keep in mind the error propagation equation $\Delta V \approx 120000\, \Delta s$, we can quickly answer other questions about measuring the same cube. For instance, suppose we wanted to determine the volume of the cube to within 5,000 cubic inches. How accurately would we have to measure the side? Thus we are given $\Delta V = 5000$, and we conclude $\Delta s \approx 5000/120000 \approx .04$ inches. This is just a little more than 1/32 inch.

Relative Error

Suppose we have a second cube whose side is twice as large ($s = 400$ inches), and once again we measure its length with an error of 1/2 inch. Then the error in the calculated value of the volume is

$$\Delta V \approx V'(400) \cdot \Delta s = 480000 \times .5 = 240{,}000 \text{ cubic inches.}$$

(In the exercises you will be asked to show $V'(400) = 480{,}000$.) The error in our calculation for the bigger cube is four times what it was for the smaller cube, even though the length was measured to the same accuracy in both cases! There is no mistake here. In fact, the volume of the second cube is eight times the volume of the first, so the numbers we are dealing with in the second case are roughly eight times as large. We should not be surprised if the error is larger, too.

Bigger numbers have bigger errors

In general, we must expect that the size of an error will depend on the size of the numbers we are working with. We expect big numbers to have big errors and small numbers to have small errors. In a sense, though, an error of 1 inch in a measurement of 50 inches is no worse than an error of 1/10-th of an inch in a measurement of 5 inches: both errors are 1/50-th the size of the quantity being measured.

A watchmaker who measures the tiny objects that go into a watch only as accurately as a carpenter measures lumber would never make a watch that worked; likewise, a carpenter who takes the pains to measure things as accurately as a watchmaker does would take forever to build a house. The scale of allowable errors is dictated by the scale of the objects they work on.

Absolute and relative error

The errors Δx we have been considering are called **absolute errors**; their values depend on the size of the quantities x we are working with. To reduce the effect of differences due to the size of x, we can look instead at the error as a *fraction* of the number being measured or calculated. This fraction $\Delta x/x$ is called **relative error**. Consider two measurements: one is 50 inches with an error of ± 1 inch; the other is 2 inches with an error of $\pm .1$ inch. The *absolute* error in the second measurement is much smaller than in the first, but the *relative* error is $2\frac{1}{2}$ times larger. (The first relative error is .02 inch per inch, the second is .05 inch per inch.) To judge how good or bad a measurement really is, we usually take the relative error instead of the absolute error.

Let's compare the propagation of relative and absolute errors. For example, the absolute error in calculating the volume of a cube whose side measures s is

$$\Delta V \approx V'(s) \cdot \Delta s.$$

The absolute errors are proportional, but the multiplier $V'(s)$ depends on the size of s. (We saw above that the multiplier is 120,000 cubic inches per inch when $s = 200$ inches, but it grows to 480,000 cubic inches per inch when $s = 400$ inches.)

In section 5, which deals with formulas for derivatives, we will see that $V'(s) = 3\,s^2$. If we substitute $3s^2$ for $V'(s)$ in the propagation equation for absolute error, we get

$$\Delta V \approx 3s^2 \cdot \Delta s.$$

To see how relative error propagates, let us divide this equation by $V = s^3$:

$$\frac{\Delta V}{V} \approx \frac{3s^2 \cdot \Delta s}{s^3} = 3\,\frac{\Delta s}{s}$$

The relative errors are proportional, but the multiplier is always 3; it doesn't depend on the size of the cube, as it did for absolute errors.

Return to the case where $\Delta s = .5$ inch and $s = 200$ inches. Since Δs and s have the same units, the relative error $\Delta s/s$ is "dimensionless"—it has no units. We can, however, describe $\Delta s/s$ as a *percentage*: $\Delta s/s = .5/200 = .25\%$, or 1/4 of 1%. For this reason, relative error is sometimes called **percentage error**. It tells us the error in measuring a quantity as a *percentage* of the value of that quantity. Since the percentage error in volume is

Percentage error is relative error

$$\frac{\Delta V}{V} = \frac{60{,}000 \text{ cu. in.}}{8{,}000{,}000 \text{ cu. in.}} = .0075 = .75\%$$

we see that the percentage error in volume is 3 times the percentage error in length—*and this is independent of the length and volumes involved*. This is what the propagation equation for relative error says: A 1% error in measuring s, whether $s = .0002$ inches or $s = 2000$ inches, will produce a 3% error in the calculated value of the volume.

Exercises

Estimation

1. a) Suppose you are going on a 110 mile trip. Then the time T it takes to make the trip is a function of how fast you drive:

$$T(v) = \frac{110 \text{ miles}}{v \text{ miles per hour}} = 110\,v^{-1} \text{ hours}\,.$$

If you drive at $v = 55$ miles per hour, T will be 2 hours. Use a computer microscope to calculate $T'(55)$ and write an English sentence interpreting this number.

b) More generally, if you and a friend are driving separate cars on a 110 mile trip, and you are travelling at some velocity v, while her speed is 1% greater than yours, then her travel time is less. How much less, as a percentage of yours? Use the formula $T'(v) = -110v^{-2}$, which can be obtained using rules given in the next section.

2. a) Suppose you have 600 square feet of plywood which you are going to use to construct a cubical box. Assuming there is no waste, what will its volume be?

b) Find a general formula which expresses the volume V of the box as a function of the area A of plywood available.

c) Use a microscope to determine $V'(600)$, and express its significance in an English sentence.

d) Use this multiplier to estimate the additional amount of plywood you would need to increase the volume of the box by 10 cubic feet.

e) In the original problem, if you had to allow for wasting 10 square feet of plywood in the construction process, by how much would this decrease the volume of the box?

f) In the original problem, if you had to allow for wasting 2% of the plywood in the construction process, by what percentage would this decrease the volume of the box?

3. Let $R(s) = 1/s$. You can use the fact that $R'(s) = -1/s^2$, to be established in section 5. Since $R(100) =$ ________ and $R'(100) =$ ________, we can make the following approximations:

$$1/97 \approx \_\_\_\_\_\_ \qquad 1/104 \approx \_\_\_\_\_\_ \qquad R(\_\_\_\_\_\_) \approx .0106\,.$$

4. Using the fact that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = 1/(2\sqrt{x})$, you can estimate the square roots of numbers that are close to perfect squares.

a) For instance $f(4) =$ ______ and $f'(4) =$ ______, so $\sqrt{4.3} \approx$ ______.

b) Use the values of $f(4)$ and $f'(4)$ to approximate $\sqrt{5}$ and $\sqrt{3.6}$.

c) Use the values of $f(100)$ and $f'(100)$to approximate $\sqrt{101}$ and $\sqrt{99.73}$.

Error analysis

5. a) If you measure the side of a square to be 12.3 inches, with an uncertainty of $\pm.05$ inch, what is your relative error?

b) What is the area of the square? Write an error propagation equation that will tell you how uncertain you should be about this value.

c) What is the relative error in your calculation of the area?

d) If you wanted to calculate the area with an error of less than 1 square inch, how accurately would you have to measure the length of the side? If you wanted the error to be less than .1 square inch, how accurately would you have to measure the side?

6. a) Suppose the side of a square measures x meters, with a possible error of Δx meters. Write the equation that describes how the error in length propagates to an error in the area. (The derivative of $f(x) = x^2$ is $f'(x) = 2x$; see section 5.)

b) Write an equation that describes how the *relative* error in length propagates to a *relative* error in area.

7. You are trying to measure the height of a building by dropping a stone off the top and seeing how long it takes to hit the ground, knowing that the distance d (in feet) an object falls is related to the time of fall, t (in seconds), by the formula $d = 16t^2$. You find that the time of fall is 2.5 seconds, and you estimate that you are accurate to within a quarter of a second. What do you calculate the height of the building to be, and how much uncertainty do you consider your calculation to have?

8. You see a flash of lightning in the distance and note that the sound of thunder arrives 5 seconds later. You know that at 20°C sound travels at 343.4 m/sec. This gives you an estimate of

$$5 \text{ sec} \times 343.4 \, \frac{\text{meters}}{\text{sec}} = 1717 \text{ meters}$$

for the distance between you and the spot where the lightning struck. You also know that the velocity v of sound varies as the square root of the temperature T measured in degrees Kelvin (the Kelvin temperature = Celsius temperature + 273), so

$$v(T) = k\sqrt{T}$$

for some constant k.

a) Use the information given here to determine the value of k.

b) If your estimate of the temperature is off by 5 degrees, how far off is your estimate of the distance to the lightning strike? How significant is this source of error likely to be in comparison with the imprecision with which you measured the 5 second time lapse? (Suppose your uncertainty about the time is .25 seconds.) Give a clear analysis justifying your answer.

9. We can measure the distance to the moon by bouncing a laser beam off a reflector placed on the moon's surface and seeing how long it takes the beam to make the round trip. If the moon is roughly 400,000 km away, and if light travels at 300,000 km/sec, how accurately do we have to be able to measure the length of the time interval to be able to determine the distance to the moon to the nearest .1 meter?

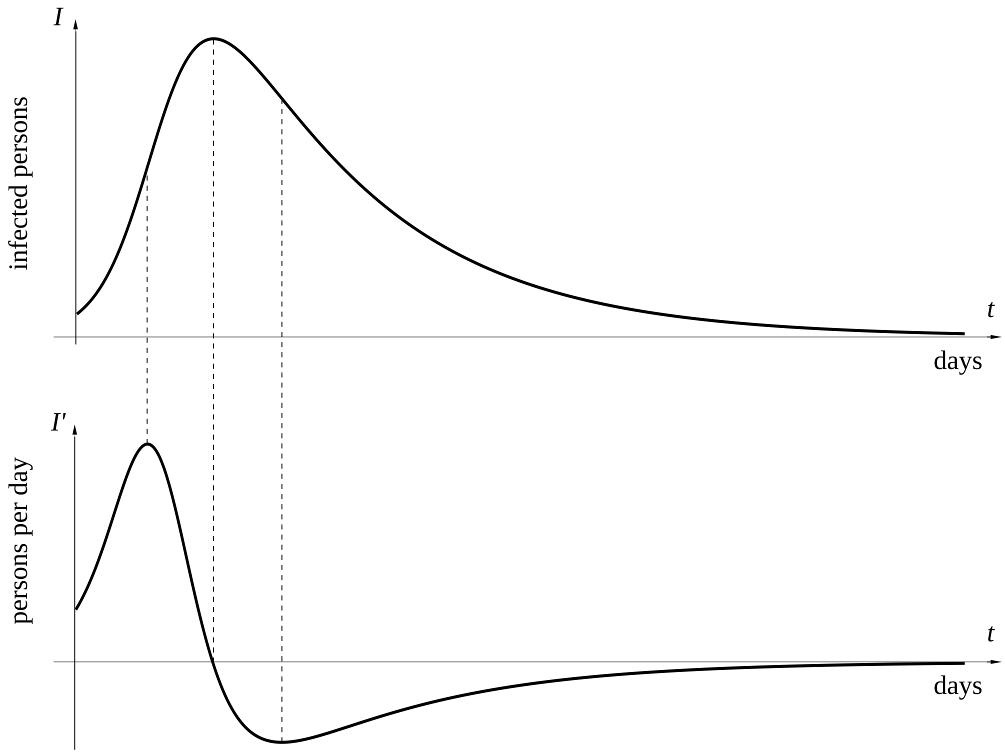
3.5 A Global View

Derivative as Function

Up to now we have looked upon the derivative as a *number*. It gives us information about a function at a *point*—the rate at which the function is changing, the slope of its graph, the value of the multiplier in the microscope equation. But the numerical value of the derivative varies from point to point, and these values can also be considered as the values of a new function—the derivative function—with its own graph. Viewed this way the derivative is a *global* object.

The derivative is a function in its own right

The connection between a function and its derivative can be seen very clearly if we look at their graphs. To illustrate, we'll use the function $I(t)$ that describes how the size of an infected population varies over time, from the S-I-R problem we analyzed in chapter 1. The graph of I appears below, and directly beneath it is the graph of I', the derivative of I. The graphs are lined-up vertically: the values of $I(a)$ and $I'(a)$ are recorded on the same vertical line that passes through the point $t = a$ on the t-axis.



The height of I' is the slope of I

To understand the connection between the graphs, keep in mind that the derivative represents a slope. Thus, at any point t, the *height* of the lower graph (I') tells us the *slope* of the upper graph (I). At the points where I is increasing, I' is positive—that is, I' lies *above* its t-axis. At the point where I is increasing most rapidly, I' reaches its highest value. In other words, where the graph of I is steepest, the graph of I' is highest. At the point where I is decreasing most rapidly, I' has its lowest value.

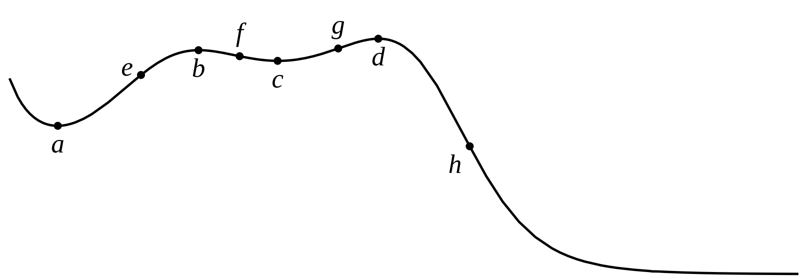
Next, consider what happens when I itself reaches its maximum value. Since I is about to switch from increasing to decreasing, the derivative must be about to switch from positive to negative. Thus, at the moment when I is largest, I' must be zero. Note that the highest point on the graph of I lines up with the point where I' crosses the t-axis. Furthermore, if we zoomed in on the graph of I at its highest point, we would find a horizontal line—in other words, one whose slope is zero.

All functions and their derivatives are related the same way that I and I' are. In the following table we list the various features of the graph of a function; alongside each is the corresponding feature of the graph of the derivative.

function	derivative
increasing	positive
decreasing	negative
horizontal	zero
steep (rising or falling)	large (positive or negative)
gradual (rising or falling)	small (positive or negative)
straight	horizontal

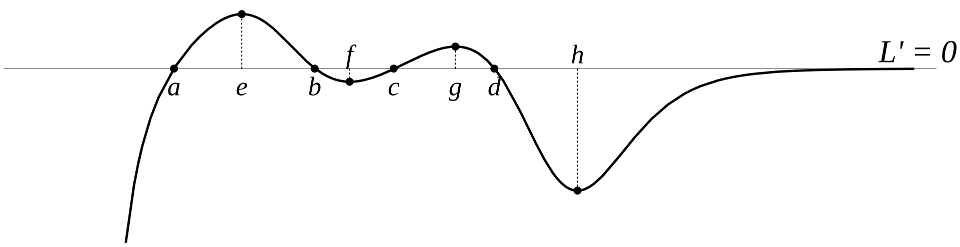
By using this table, you should be able to make a rough sketch of the graph of the derivative, when you are given the graph of a function. You can also read the table from right to left, to see how the graph of a function is influenced by the graph of its derivative.

For instance, suppose the graph of the function $L(x)$ is

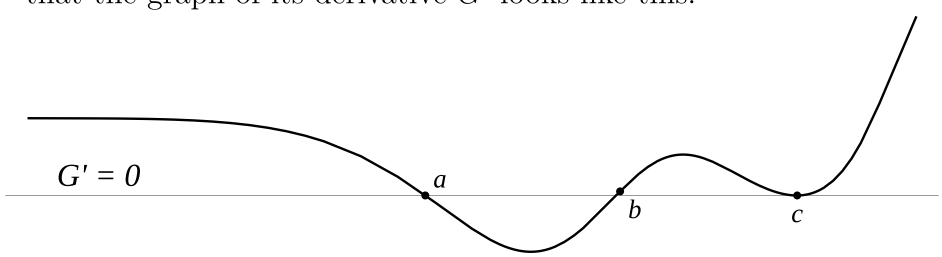


Then we know that its derivative L' must be 0 at points a, b, c, and d; that the derivative must be positive between a and b and between c and d, negative otherwise; that the derivative takes on relatively large values at e and g (positive) and at f and h (negative); that the derivative must approach 0 at the right endpoint and be large and negative at the left endpoint. Putting all this together we conclude that the graph of the derivative L' must look something like following:

Finding a derivative "by eye"



Conversely, suppose all we are told about a certain function G is that the graph of its derivative G' looks like this:



Then we can infer that the function G itself is decreasing between a and b and is increasing everywhere else; that the graph of G is horizontal at a, b, and c; that both ends of the graph of G slope upward from left to right—the left end more or less straight, the right getting steeper and steeper.

Formulas for Derivatives

Basic Functions

If a function is given by a formula, then its derivative also has a formula, defined for the points where the function is locally linear. The formula is produced by a definite process, called **differentiation**. In this section we look at some of the basic aspects, and in the next we will take up the chain rule, which is the key to the whole process. Then in chapter 5 we will review differentiation systematically.

Formulas are combinations of basic functions

Most formulas are constructed by combining only a few basic functions in various ways. For instance, the formula

$$3\,x^7 - \frac{\sin x}{8\sqrt{x}},$$

uses the basic functions x^7, $\sin x$, and $\sqrt{x}$. In fact, since $\sqrt{x} = x^{1/2}$, we can think of x^7 and $\sqrt{x}$ as two different instances of a single basic "power of x"—which we can write as x^p.

The following table lists some of the more common basic functions with their derivatives. The number c is an arbitrary constant, and so is the power p. The last function in the table is the exponential function with base b. Its derivative involves a parameter k_b that varies with the base b. For instance, exercise 6 in section 3 established that, when $b = 2$, then $k_2 \approx .69$. Exercise 7 established, for any base b, that k_b is the value of the derivative of b^x when $x = 0$. We will have more to say about the parameter k_b in the next chapter.

function	derivative
c	0
x^p	px^{p-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
b^x	$k_b \cdot b^x$

Remember that the input to the trigonometric functions is always measured in radians; the above formulas are not correct if x is measured in degrees. There are similar formulas if you insist on using degrees, but they are more complicated. This is the principal reasons we work in radians—the formulas are nice!

For example,

- the derivative of $1/x = x^{-1}$ is $-x^{-2} = -1/x^2$;
- the derivative of $\sqrt{w} = w^{1/2}$ is $\frac{1}{2}w^{-1/2} = \dfrac{1}{2\sqrt{w}}$;
- the derivative of x^π is $\pi x^{\pi-1}$; and
- the derivative of π^x is $k_\pi \cdot \pi^x \approx 1.14\,\pi^x$.

Compare the last two functions. The first, x^π, is a **power function**—it is a power of the input x. The second, π^x, is an **exponential function**—the input x appears in the exponent. When you differentiate a power function, the exponent drops by 1; when you differentiate an exponential function, the exponent doesn't change.

Basic Rules

Since basic functions are combined in various ways to make formulas, we need to know how to differentiate *combinations*. For example, suppose we add the basic functions $g(x)$ and $h(x)$, to get $f(x) = g(x) + h(x)$. Then f is differentiable, and $f'(x) = g'(x) + h'(x)$. Actually, this is true for *all* differentiable functions g and h, not just basic functions. It says: "The rate at which f changes is the sum of the separate rates at which g and h change." Here are some examples that illustrate the point.

The addition rule

$$\text{If} \quad f(x) = \tan x + x^{-6}, \quad \text{then} \quad f'(x) = \sec^2 x - 6x^{-7}.$$

$$\text{If} \quad f(w) = 2^w + \sqrt{w}, \quad \text{then} \quad f'(w) = k_2\, 2^w + \frac{1}{2\sqrt{w}} \text{ (and } k_2 \approx .69).$$

Likewise, if we multiply any differentiable function g by a constant c, then the product $f(x) = cg(x)$ is also differentiable and $f'(x) = cg'(x)$. This says: "If f is c times as large as g, then f changes at c times the rate of g." Thus the derivative of $5 \sin x$ is $5 \cos x$. Likewise, the derivative of $(5x)^2$ is $50\,x$. (This took an extra calculation.) However, the rule does *not* tell us how to find the derivative of $\sin(5x)$, because $\sin(5x) \neq 5\sin(x)$. We will need the chain rule to work this one out.

The constant multiple rule

The rules about sums and constant multiples of functions are just the first of several basic rules for differentiating combinations of functions. We will describe how to handle products and quotients of functions in chapter 5. For the moment we summarize in the following table the rules we have already covered.

function	derivative
$f(x) + g(x)$	$f'(x) + g'(x)$
$c \cdot f(x)$	$c \cdot f'(x)$

With just the few facts already laid out we can differentiate a variety of functions given by formulas. In particular, we can differentiate any **polyno-**

mial function:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here a_n, a_{n-1}, ..., a_2, a_1, a_0 are various constants, and n is a positive integer, called the **degree** of the polynomial. A polynomial is a sum of constant multiples of integer powers of the input variable. A polynomial of degree 1 is just a linear function. The derivative is

$$P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1.$$

All the rules presented up to this point are illustrated in the following examples; note that the first three involve polynomials.

function	derivative
$7x + 2$	7
$5x^4 - 2x^3$	$20x^3 - 6x^2$
$5x^4 - 2x^3 + 17$	$20x^3 - 6x^2$
$3u^{15} + .5u^8 - \pi u^3 + u - \sqrt{2}$	$45u^{14} + 4u^7 - 3\pi u^2 + 1$
$6 \cdot 10^z + 17/z^5$	$6 \cdot k_{10}\, 10^z - 85/z^6$
$3 \sin t - 2t^3$	$3 \cos t - 6t^2$
$\pi \cos x - \sqrt{3} \tan x + \pi^2$	$-\pi \sin x - \sqrt{3} \sec^2 x$

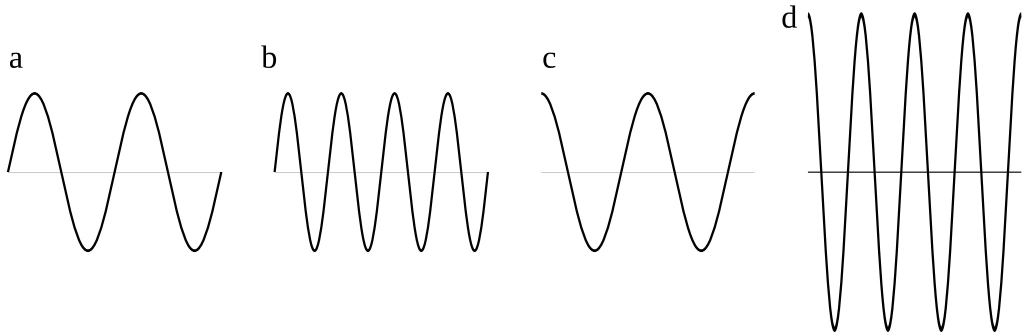
The first two functions have the same derivative because they differ only by a constant, and the derivative of a constant is zero. The constant k_{10} that appears in the fourth example is approximately 2.30.

Exercises

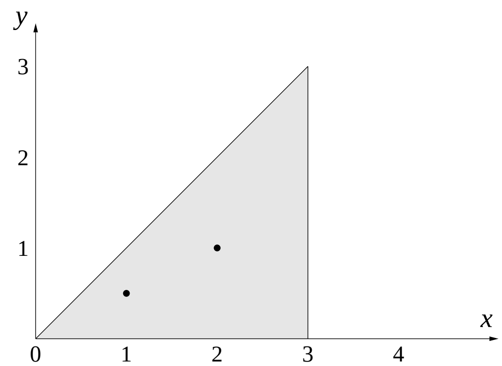
Sketching the graph of the derivative

1. Sketch the graphs of two different functions that have the same derivative. (For example, can you find two *linear* functions that have the same derivative?)

2. Here are the graphs of four related functions: s, its derivative s', another function $c(t) = s(2t)$, and *its* derivative $c'(t)$. The graphs are out of order. Label them with the correct names s, s', c, and c'.



3. a) Suppose a function $y = g(x)$ satisfies $g(0) = 0$ and $0 \leq g'(x) \leq 1$ for all values of x in the interval $0 \leq x \leq 3$. Explain carefully why the graph of g must lie entirely in the triangular region shaded below:



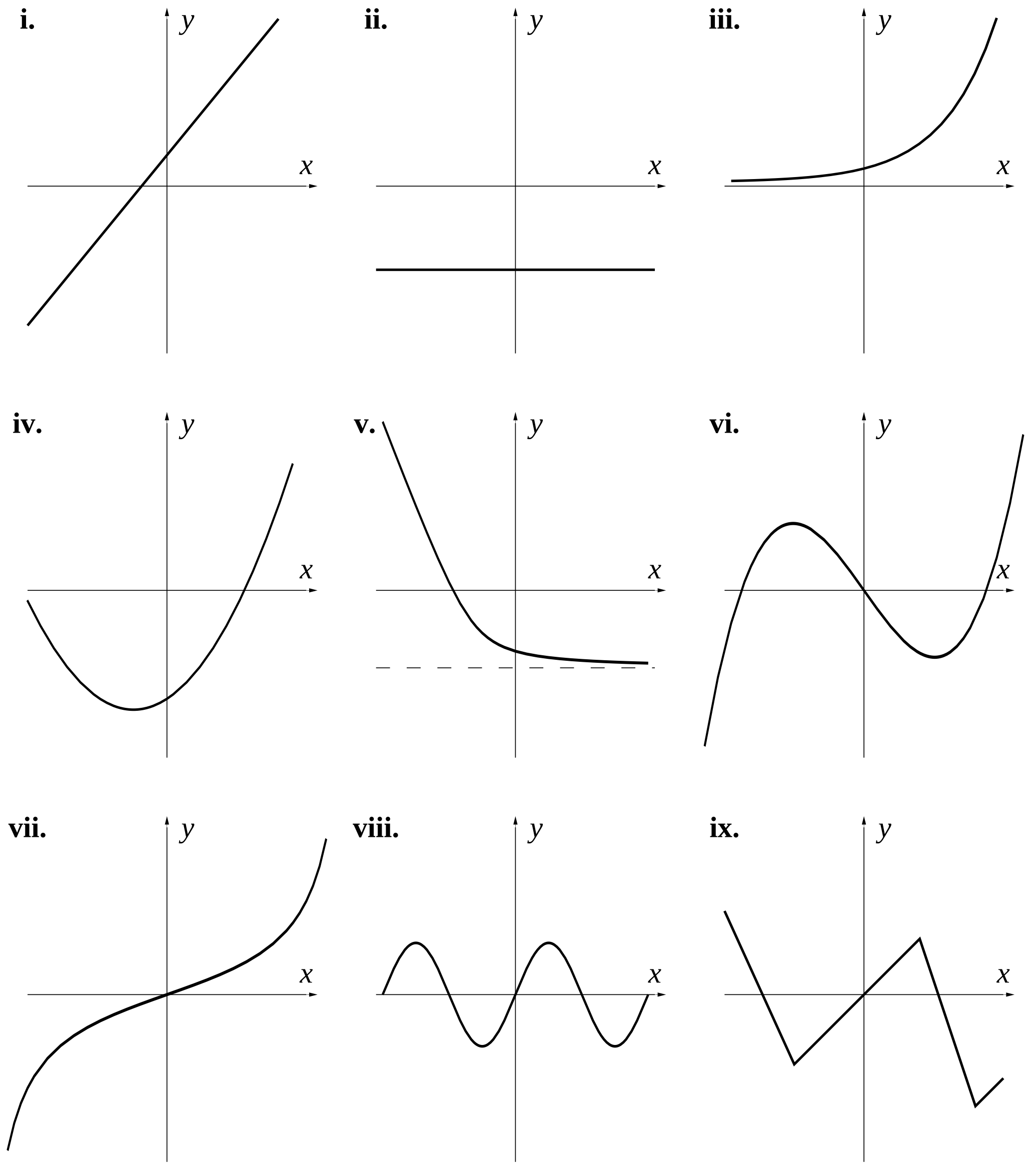
b) Suppose you learn that $g(1) = .5$ and $g(2) = 1$. Draw the smallest shaded region in which you can guarantee that the graph of g must lie.

4. Suppose h is differentiable over the interval $0 \leq x \leq 3$. Suppose $h(0) = 0$, and that

$$\begin{aligned} .5 \leq h'(x) \leq 1 &\quad \text{for} \quad 0 \leq x \leq 1 \\ 0 \leq h'(x) \leq .5 &\quad \text{for} \quad 1 \leq x \leq 2 \\ -1 \leq h'(x) \leq 0 &\quad \text{for} \quad 2 \leq x \leq 3 \end{aligned}$$

Draw the smallest shaded region in the x, y-plane in which you can guarantee that the graph of $y = h(x)$ must lie.

5. For each of the functions graphed below, sketch the graph of its derivative.



Differentiation

6. Find formulas for the derivatives of the following functions; that is, *differentiate* them.

a) $f(x) = 3x^7 - .3x^4 + \pi x^3 - 17$

b) $g(x) = \sqrt{3}\,\sqrt{x} + \dfrac{7}{x^5}$

c) $h(w) = 2w^8 - \sin w + \dfrac{1}{3w^2}$

d) $R(u) = 4\cos u - 3\tan u + \sqrt[3]{u}$

e) $V(s) = \sqrt[4]{16} - \sqrt[4]{s}$

f) $F(z) = \sqrt{7} \cdot 2^z + (1/2)^z$

g) $P(t) = -\dfrac{a}{2}t^2 + v_0 t + d_0$ (a, v_0, and d_0 are constants)

7. Use a computer graphing utility for this exercise. Graph on the same screen the following three functions:

1. the function f given below, on the indicated interval;
2. the function $g(x) = (f(x + .01) - f(x - .01))/.02$ that estimates the slope of the graph of f at x;
3. the function $h(x) = f'(x)$, where you use the differentiation rules to find f'.

a) $f(x) = x^4$ on $-1 \le x \le 1$.

b) $f(x) = x^{-1}$ on $1 \le x \le 8$.

c) $f(x) = \sqrt{x}$ on $.25 \le x \le 9$.

d) $f(x) = \sin x$ on $0 \le x \le 2\pi$.

The graphs of g and h should coincide—or "share phosphor"—in each case. Do they?

8. In each case below, find a function $f(x)$ whose derivative $f'(x)$ is:

a) $f'(x) = 12\,x^{11}$.

b) $f'(x) = 5x^7$.

c) $f'(x) = \cos x + \sin x$.

d) $f'(x) = ax^2 + bx + c$.

e) $f'(x) = 0$.

f) $f'(x) = \dfrac{5}{\sqrt{x}}$.

9. What is the slope of the graph of $y = x - \sqrt{x}$ at $x = 4$? At $x = 100$? At $x = 10000$?

10. a) For which values of x is the function $x - x^3$ increasing?

b) Where is the graph of $y = x - x^3$ rising most steeply?

c) At what points is the graph of $y = x - x^3$ horizontal?

d) Make a sketch of the graph of $y = x - x^3$ that reflects all these results.

11. a) Sketch the graph of the function $y = 2x + \dfrac{5}{x}$ on the interval $.2 \leq x \leq 4$.

b) Where is the lowest point on that graph? Give the value of the x-coordinate *exactly*. [Answer: $x = \sqrt{5/2}$.]

12. What is the slope of the graph of $y = \sin x + \cos x$ at $x = \pi/4$?

13. a) Write the microscope equation for $y = \sin x$ at $x = 0$.

b) Using the microscope equation, estimate the following values: $\sin .3$, $\sin .007$, $\sin(-.02)$. Check these values with a calculator. (Remember to set your calculator to radian mode!)

14. a) Write the microscope equation for $y = \tan x$ at $x = 0$.

b) Estimate the following values: $\tan .007$, $\tan .3$, $\tan(-.02)$. Check these values with a calculator.

15. a) Write the microscope equation for $y = \sqrt{x}$ at $x = 3600$.

b) Use the microscope equation to estimate $\sqrt{3628}$ and $\sqrt{3592}$. How far are these estimates from the values given by a calculator?

16. If the radius of a spherical balloon is r inches, its volume is $\frac{4}{3}\pi r^3$ cubic inches.

a) At what rate does the volume increase, in cubic inches per inch, when the radius is 4 inches?

b) Write the microscope equation for the volume when $r = 4$ inches.

c) When the radius is 4 inches, approximately how much does it increase if the volume is increased by 50 cubic inches?

d) Suppose someone is inflating the balloon at the rate of 10 cubic inches of air per second. If the radius is 4 inches, at what rate is it increasing, in inches per second?

17. A ball is held motionless and then dropped from the top of a 200 foot tall building. After t seconds have passed, the distance from the ground to the ball is $d = f(t) = -16t^2 + 200$ feet.

a) Find a formula for the velocity $v = f'(t)$ of the ball after t seconds. Check that your formula agrees with the given information that the initial velocity of the ball is 0 feet/second.

b) Draw graphs of both the velocity and the distance as functions of time. What time interval makes physical sense in this situation? (For example, does $t < 0$ make sense? Does the distance formula make sense after the ball hits the ground?)

c) At what time does the ball hit the ground? What is its velocity then?

18. A second ball is tossed straight up from the top of the same building with a velocity of 10 feet per second. After t seconds have passed, the distance from the ground to the ball is $d = f(t) = -16t^2 + 10t + 200$ feet.

a) Find a formula for the velocity of the second ball. Does the formula agree with given information that the initial velocity is +10 feet per second? Compare the velocity formulas for the two balls; how are they similar, and how are they different?

b) Draw graphs of both the velocity and the distance as functions of time. What time interval makes physical sense in this situation?

c) Use your graph to answer the following questions. During what period of time is the ball rising? During what period of time is it falling? When does it reach the highest point of its flight?

d) How high does the ball rise?

19. a) What is the velocity formula for a third ball that is thrown *downward* from the top of the building with a velocity of 40 feet per second? Check that your formula gives the correct initial velocity.

b) What is the distance formula for the third ball? Check that it satisfies the initial condition (namely, that the ball starts at the top of the building).

c) When does this ball hit the ground? How fast is it going then?

20. A steel ball is rolling along a 20-inch long straight track so that its distance from the midpoint of the track (which is 10 inches from either end) is $d = 3\sin t$ inches after t seconds have passed. (Think of the track as aligned from left to right. Positive distances mean the ball is to the right of the center; negative distances mean it is to the left.)

a) Find a formula for the velocity of the ball after t seconds. What is happening when the velocity is positive; when it is negative; when it equals zero? Write a sentence or two describing the motion of the ball.

b) How far from the midpoint of the track does the ball get? How can you tell?

c) How fast is the ball going when it is at the midpoint of the track? Does it ever go faster than this? How can you tell?

21. A forester who wants to know the height of a tree walks 100 feet from its base, sights to the top of the tree, and finds the resulting angle to be 57 degrees.

a) What height does this give for the tree?

b) If the measurement of the angle is certain only to 5 degrees, what can you say about the uncertainty of the height found in part (a)? (Note: you need to express angles in *radians* to use the formulas from calculus: π radians = 180 degrees.)

22. a) In the preceding problem, what percentage error in the height of the tree is produced by a 1 degree error in measuring the angle?

b) What would the percentage error have been if the angle had been 75 degrees instead of 57 degrees? 40 degrees?

c) If you can measure angles to within 1 degree accuracy and you want to measure the height of a tree that's roughly 150 feet tall by means of the technique in the preceding problem, how far away from the tree should you stand to get your best estimate of the tree's height? How accurate would your answer be?

3.6 The Chain Rule

Combining Rates of Change

Let's return to the expanding house that we studied in section 4. When the temperature T increased, every side s of the house got longer; when s got longer, the volume V got larger. We already discussed how V responds to changes in s, but that's only part of the story. What we'd really like to know is this: exactly how does the volume V respond to changes in temperature T? We can work this out in stages: first we see how V responds to changes in s, and then how s responds to changes in T.

Stage 1. Our "house" is a cube that measures 200 inches on a side, and the microscope equation (section 4) describes how V responds to changes in s:

How volume responds to changes in length

$$\Delta V \approx 120000 \frac{\text{cubic inches of volume}}{\text{inch of length}} \cdot \Delta s \text{ inches.}$$

Stage 2. Physical experiments with wood show that a 200 inch length of wood increases about .0004 inches in length per degree Fahrenheit. This is a *rate*, and we can build a second microscope equation with it:

How length responds to changes in temperature

$$\Delta s \approx .0004 \frac{\text{inches of length}}{\text{degree F}} \cdot \Delta T \text{ degrees F,}$$

where ΔT measures the change in temperature, in degrees Fahrenheit.

We can combine the two stages because Δs appears in both. Replace Δs in the first equation by the right-hand side of the second equation. The result is

$$\Delta V \approx 120000 \frac{\text{cubic inches}}{\text{inch}} \times .0004 \frac{\text{inches}}{\text{degree F}} \cdot \Delta T \text{ degrees F.}$$

We can condense this to

How volume responds to changes in temperature

$$\Delta V \approx 48 \frac{\text{cubic inches}}{\text{degree F}} \cdot \Delta T \text{ degrees F.}$$

This is a *third* microscope equation, and it shows directly how the volume of the house responds to changes in temperature. It is the answer to our question.

As always, the multiplier in a microscope equation is a rate. The multiplier in the third microscope equation, 48 cubic inches/degree F, tells us the rate at which *volume changes with respect to temperature*. Thus, if the temperature increases by 10 degrees between night and day, the house will become about 480 cubic inches larger. Recall that Bodanis (see section 4) said that the house might become only a few cubic inches larger—say, $\Delta V = 3$ cubic inches. If we solve the microscope equation

$$3 \approx 48 \cdot \Delta T$$

for ΔT, we see that the temperature would have risen only 1/16-th of a degree F!

The rate that appears as the multiplier in the third microscope equation is the product of the other two:

How the rates combine

$$48 \ \frac{\text{cubic inches}}{\text{degree F}} = 120000 \ \frac{\text{cubic inches}}{\text{inch}} \times .0004 \ \frac{\text{inches}}{\text{degree F}}.$$

Each of these rates is a derivative:

$$\underbrace{48 \ \frac{\text{cubic inches}}{\text{degree F}}}_{dV/dT} = \underbrace{120000 \ \frac{\text{cubic inches}}{\text{inch}}}_{dV/ds} \times \underbrace{.0004 \ \frac{\text{inches}}{\text{degree F}}}_{ds/dT}.$$

We wrote the derivatives in Leibniz's notation because it's particularly helpful in keeping straight what is going on. For instance, dV/dT indicates very clearly the rate at which volume is changing with respect to *temperature*, and dV/ds the rate at which it is changing with respect to *length*. These rates are quite different—they even have different units—but the notation V' does not distinguish between them. In Leibniz's notation, the relation between the three rates takes this striking form:

$$\frac{dV}{dT} = \frac{dV}{ds} \cdot \frac{ds}{dT}.$$

This relation is called the **chain rule** for the variables T, s, and V. (We'll see in a moment what this has to do with chains.)

The chain rule is a consequence of the way the three microscope equations are related to each other. We can see how it emerges directly from the microscope equations if we replace the numbers that appear as multipliers in those equations by the three derivatives. To begin, we write

$$\Delta V \approx \frac{dV}{ds} \cdot \Delta s \qquad \text{and} \qquad \Delta s \approx \frac{ds}{dT} \cdot \Delta T.$$

Then, combining these equations, we get

$$\Delta V \approx \frac{dV}{ds} \cdot \frac{ds}{dT} \cdot \Delta T.$$

In fact, this is the microscope equation for V in terms of T, which can be written more directly as

$$\Delta V \approx \frac{dV}{dT} \cdot \Delta T.$$

In these two expressions we have the same microscope equation, so the multipliers must be equal. Thus, we recover the chain rule:

$$\frac{dV}{ds} \cdot \frac{ds}{dT} = \frac{dV}{dT}.$$

Recall that Leibniz worked directly with *differentials*, like dV and ds, so a derivative was a genuine fraction. For him, the chain rule is true simply because we can cancel the two appearances of "ds" in the derivatives. For us, though, a derivative is not really a fraction, so we need an argument like the one in the text to establish the rule.

Chains and the Chain Rule

Let's analyze the relationships between the three variables in the expanding house problem in more detail. There are three functions involved: volume is a function of length: $V = V(s)$; length is a function of temperature: $s = s(T)$; and finally, volume is a function of temperature, too: $V = V(s(T))$. To visualize these relationships better, we introduce the notion of an **input–output diagram**. The input–output diagram for the function $s = s(T)$ is just $T \to s$. It indicates that T is the input of a function whose output is s. Likewise $s \to V$ says that volume V is a function of length s. Since the output of $T \to s$ is the input of $s \to V$, we can make a *chain* of these two diagrams:

$$T \longrightarrow s \longrightarrow V.$$

An input–output chain

The result describes a function that has input T and output V. It is thus an input–output diagram for the third function $V = V(s(T))$.

We could also write the input–output diagram for the third function simply as $T \to V$; in other words,

$$T \longrightarrow V \qquad \text{equals} \qquad T \longrightarrow s \longrightarrow V.$$

A chain and its links

We say that $T \to s \to V$ is a **chain** that is made up of the two **links** $T \to s$ and $s \to V$. Since each input–output diagram represents a function, we can attach a derivative that describes the rate of change of the output with respect to the input:

$$T \xrightarrow{\frac{ds}{dT}} s \qquad s \xrightarrow{\frac{dV}{ds}} V \qquad T \xrightarrow{\frac{dV}{dT}} V$$

Here is a single picture that shows all the relationships:

$$\overset{\frac{dV}{dT}}{\overbrace{T \underset{\frac{ds}{dT}}{\longrightarrow} s \underset{\frac{dV}{ds}}{\longrightarrow} V}}$$

We can thus relate the derivative dV/dT of the whole chain to the derivatives dV/ds and ds/dT of the individual links by

$$\frac{dV}{dT} = \frac{dV}{ds} \cdot \frac{ds}{dT}.$$

The same argument holds for any chain of functions. If u is a function of x, and if y is some function of u, then a small change in x produces a small change in u and hence in y. The total multiplier for the chain is simply the product of the multipliers of the individual links:

The chain rule: $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Moreover, an obvious generalization extends this result to a chain containing more than two links.

A simple example. We can sometimes use the chain rule without giving it much thought. For instance, suppose a bookstore makes an average profit of \$3 per book, and its sales are increasing at the rate of 40 books per month. At what rate is its monthly profit increasing, in dollars per month? Does it seem clear to you that the rate is \$120 per month?

Let's analyze the question in more detail. There are three variables here:

time	t	measured in months;
sales	s	measured in books;
profit	p	measured in dollars.

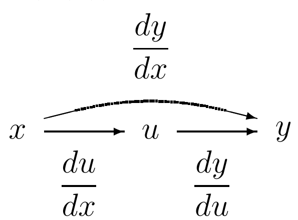
The two known rates are

$$\frac{dp}{ds} = 3\,\frac{\text{dollars}}{\text{book}} \qquad \text{and} \qquad \frac{ds}{dt} = 40\,\frac{\text{books}}{\text{month}}.$$

The rate we seek is dp/dt, and we find it by the chain rule:

$$\begin{aligned}\frac{dp}{dt} &= \frac{dp}{ds}\cdot\frac{ds}{dt}\\ &= 3\,\frac{\text{dollars}}{\text{book}} \times 40\,\frac{\text{books}}{\text{month}}\\ &= 120\,\frac{\text{dollars}}{\text{month}}\end{aligned}$$

Chains, in general. The chain rule applies whenever the output of one function is the input of another. For example, suppose $u = f(x)$ and $y = g(u)$. Then $y = g(f(x))$, and we have:



$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Let's take

$$u = x^2 \qquad \text{and} \qquad y = \sin(u);$$

then $y = \sin(x^2)$, and it is not at all obvious what the derivative dy/dx ought to be. None of the basic rules in section 4 covers this function. However, those rules *do* cover $u = x^2$ and $y = \sin(u)$:

$$\frac{du}{dx} = 2x \qquad \text{and} \qquad \frac{dy}{du} = \cos(u).$$

We can now get dy/dx by the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \cos(u)\cdot 2x.$$

Since we are interested in y as a function of x—rather than u—we should rewrite dy/dx so that it is expressed entirely in terms of x:

$$\text{If} \quad y = \sin(x^2), \quad \text{then} \quad \frac{dy}{dx} = 2x\cos(u) = 2x\cos(x^2).$$

Let's start over, using the function names f and g we introduced at the outset:

$$u = f(x) \quad \text{and} \quad y = g(u), \quad \text{so} \quad y = g(f(x)).$$

The third function, $y = g(f(x))$, needs a name of its own; let's call it h. Thus

Composition of functions

$$y = h(x) = g(f(x)).$$

We say that h is **composed** of g and f, and h is called the **composite**, or the **composition**, of g and f.

The problem is to find the derivative h' of the composite function, knowing g' and f'. Let's translate all the derivatives into Leibniz's notation.

$$h'(x) = \frac{dy}{dx} \qquad g'(u) = \frac{dy}{du} \qquad f'(x) = \frac{du}{dx}.$$

We can now invoke the chain rule:

$$h'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = g'(u) \cdot f'(x).$$

Although h' is now expressed in terms of g' and f', we are not yet done. The variable u that appears in $g'(u)$ is out of place—because h is a function of x, not u. (We got to the same point in the example; the original form of the derivative of $\sin(x^2)$ was $2x\cos(u)$.) The remedy is to replace u by $f(x)$; we can do this because $u = f(x)$ is given.

The chain rule: $h'(x) = g'(f(x)) \cdot f'(x)$ **when** $h(x) = g(f(x))$

There is a certain danger in a formula as terse and compact as this that it loses all conceptual meaning and becomes simply a formal string of symbols to be manipulated blindly. You should always remember that the expression in the box is just a mathematical statement of the intuitively clear idea that when two functions are chained together, with the output of one serving as the input of the other, then the combined function has a multiplier which is simply the product of the multipliers of the two constituent functions.

Using the Chain Rule

The chain rule will allow us to differentiate nearly any formula. The key is to recognize when a given formula can be written as a chain—and then, how to write it.

Example 1. Here is a problem first mentioned on page 149: What is the derivative of $y = \sin(5x)$? If we set

$$y = \sin(u) \qquad \text{where} \qquad u = 5x,$$

then we find immediately

$$\frac{dy}{du} = \cos(u) \qquad \text{and} \qquad \frac{du}{dx} = 5.$$

Thus, by the chain rule we see

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u) \cdot 5 = 5\cos(5x).$$

Example 2. $w = 2^{\cos z}$. Set

$$w = 2^u \qquad \text{and} \qquad u = \cos z.$$

Then, once again, the basic rules from section 4 are sufficient to differentiate the individual links:

$$\frac{dw}{du} = k_2\, 2^u \qquad \text{and} \qquad \frac{du}{dz} = -\sin z.$$

The chain rule does the rest:

$$\frac{dw}{dz} = \frac{dw}{du} \cdot \frac{du}{dz} = k_2\, 2^u \cdot (-\sin z) = -k_2 \sin z 2^{\cos z}.$$

Example 3. $p = \sqrt{7t^3 + \sin^2 t}$. This presents several challenges. First let's make a chain:

$$p = \sqrt{u} \qquad \text{where} \qquad u = 7t^3 + \sin^2 t.$$

The basic rules give us $dp/du = 1/2\sqrt{u}$, but it is more difficult to deal with u. Let's at least introduce separate labels for the two terms in u:

$$q = 7t^3 \qquad \text{and} \qquad r = \sin^2 t.$$

Then

$$\frac{du}{dt} = \frac{dq}{dt} + \frac{dr}{dt} \qquad \text{and} \qquad \frac{dq}{dt} = 21t^2.$$

The remaining term $r = \sin^2 t = (\sin t)^2$ can itself be differentiated by the chain rule. Set

$$r = v^2 \qquad \text{where} \qquad v = \sin t.$$

Then

$$\frac{dr}{dv} = 2v \qquad \text{and} \qquad \frac{dv}{dt} = \cos(t),$$

so

$$\frac{dr}{dt} = \frac{dr}{dv} \cdot \frac{dv}{dt} = 2v \cos t = 2 \sin t \cos t.$$

The final step is to assemble all the pieces:

$$\frac{dp}{dt} = \frac{dp}{du} \cdot \frac{du}{dt} = \frac{1}{2\sqrt{u}} \cdot \left(21t^2 + 2 \sin t \cos t\right) = \frac{21t^2 + 2 \sin t \cos t}{2\sqrt{7t^3 + \sin^2 t}}$$

By breaking down a complicated expression into simple pieces, and applying the appropriate differentiation rule to each piece, it is possible to differentiate a vast array of formulas. You may meet two sorts of difficulties: you may not see how to break down the expression into simpler parts; and you may overlook a step. Practice helps overcome the first, and vigilance the second.

Here is an example of the second problem: find the derivative of $y = -3\cos(2x)$. The derivative is *not* $3\sin(2x)$; it is $6\sin(2x)$. Besides remembering to deal with the constant multiplier -3, and with the fact that there is a minus sign in the derivative of $\cos u$, you must not overlook the link $u = 2x$ in the chain that connects y to x.

Exercises

1. Use the chain rule to find dy/dx, when y is given as a function of x in the following way.

a) $y = 5u - 3$, where $u = 4 - 7x$.

b) $y = \sin u$, where $u = 4 - 7x$.

c) $y = \tan u$, where $u = x^3$.

d) $y = 10^u$, where $u = x^2$.

e) $y = u^4$, where $u = x^3 + 5$.

2. Find the derivatives of the following functions.

a) $F(x) = (9x + 6x^3)^5$.

b) $G(w) = \sqrt{4w^2 + 1}$.

c) $S(w) = \sqrt{(4w^2 + 1)^3}$.

d) $R(x) = \dfrac{1}{1-x}$. (Hint: think of $\dfrac{1}{1-x}$ as $(1-x)^{-1}$.)

e) $D(z) = 3\tan\left(\dfrac{1}{z}\right)$.

f) $\mathrm{dog}(w) = \sin^2(w^3 + 1)$.

g) $\mathrm{pig}(t) = \cos(2^t)$.

h) $\mathrm{wombat}(x) = 5^{1/x}$.

3. If $h(x) = (f(x))^6$ where f is some function satisfying $f(93) = 2$ and $f'(93) = -4$, what is $h'(93)$?

4. If $H(x) = F(x^2 - 4x + 2)$ where F is some function satisfying $F'(2) = 3$, what is $H'(4)$?

5. If $f(x) = (1 + x^2)^5$, what are the numerical values of $f'(0)$ and $f'(1)$?

6. If $h(t) = \cos(\sin t)$, what are the numerical values of $h'(0)$ and $h'(\pi)$?

7. If $f'(x) = g(x)$, which of the following defines a function which also must have g as its derivative?

$$f(x+17) \qquad f(17x) \qquad 17f(x) \qquad 17 + f(x) \qquad f(17)$$

8. Let $f(t) = t^2 + 2t$ and $g(t) = 5t^3 - 3$. Determine all of the following: $f'(t)$, $g'(t)$, $g(f(t))$, $f(g(t))$, $g'(f(t))$, $f'(g(t))$, $(f(g(t)))'$, $(g(f(t)))'$.

9. a) What is the derivative of $f(x) = 2^{-x^2}$?

b) Sketch the graphs of f and its derivative on the interval $-2 \le x \le 2$.

c) For what values(s) of x is $f'(x) = 0$? What is true about the graph of f at the corresponding points?

d) Where does the graph of f have positive slope, and where does it have negative slope?

10. a) With a graphing utility, find the point x where the function $y = 1/(3x^2 - 5x + 7)$ takes its maximum value. Obtain the numerical value of x accurately to two decimal places.

b) Find the derivative of $y = 1/(3x^2 - 5x + 7)$, and determine where it takes the value 0.

[Answer: $y' = -(6x - 5)(3x^2 - 5x + 7)^{-2}$, and $y' = 0$ when $x = 5/6$.]

c) Using part (b), find the *exact* value of x where $y = 1/(3x^2 - 5x + 7)$ takes its maximum value.

d) At what point is the graph of $y = 1/(3x^2 - 5x + 7)$ rising most steeply? Describe how you determined the location of this point.

11. a) Write the microscope equation for the function $y = \sin\sqrt{x}$ at $x = 1$.

b) Using the microscope equation, estimate the following values: $\sin\sqrt{1.05}$, $\sin\sqrt{.9}$.

12. a) Write the microscope equation for $w = \sqrt{1 + x}$ at $x = 0$.

b) Use the microscope equation to estimate the values of $\sqrt{1.1056}$ and $\sqrt{.9788}$. Compare your estimates with the values provided by a calculator.

13. When the sides of a cube are 5 inches, its surface area is changing at the rate of 60 square inches per inch increase in the side. If, at that moment, the sides are increasing at a rate of 3 inches per hour, at what rate is the area increasing: is it 60, 3, 63, 20, 180, 5, or 15 square inches per hour?

14. Find a function $f(x)$ for which $f'(x) = 3x^2(5 + x^3)^{10}$. Find a function $p(x)$ for which $p'(x) = x^2(5 + x^3)^{10}$. A useful way to proceed is to guess. For instance, you might guess $f(x) = (5 + x^3)^{11}$. While this guess isn't correct, it suggest what modification you might make to get the answer.

15. Find a function $g(t)$ for which $g'(t) = t/\sqrt{1 + t^2}$.

3.7 Partial Derivatives

Let's return to the sunrise function once again. The time of sunrise depends not only on the date, but on our latitude. In fact, if we are far enough north or south, there are days when the sun never rises at all. We give in the table below the time of sunrise at eight different latitudes on March 15, 1990.

Latitude	36°N	38°N	40°N	42°N	44°N	46°N	48°N	50°N
Mar 15	6:10	6:11	6:12	6:13	6:13	6:13	6:14	6:14

Thus on March 15, the time of sunrise increases as latitude increases.

The time of sunrise depends on latitude as well as on the date

Clearly what this shows is that the time of sunrise is actually a function of two independent inputs: the date and the latitude. If T denotes the time of sunrise, then we will write $T = T(d, \lambda)$ to make explicit the dependence of T on both the date d and the latitude λ. To capture this double dependence, we need information like the following table:

Latitude	36°N	38°N	40°N	**42°N**	44°N	46°N	48°N	50°N
Mar 3	6:24	6:27	6:31	**6:33**	6:34	6:36	6:38	6:40
7	6:20	6:22	6:25	**6:26**	6:27	6:29	6:30	6:32
11	6:15	6:17	6:19	**6:19**	6:20	6:21	6:22	6:23
15	**6:10**	**6:11**	**6:12**	**6:13**	**6:13**	**6:13**	**6:14**	**6:14**
19	6:06	6:06	6:06	**6:06**	6:06	6:06	6:06	6:06
23	6:01	6:00	5:59	**5:59**	5:58	5:58	5:58	5:57
27	5:56	5:54	5:53	**5:52**	5:51	5:50	5:49	5:48

Thus we can say $T(74, 42°\text{N}) = 6{:}13$ (March 15 is the 74-th day of the year). Note, though, that at this date and place the time of sunrise is changing in two very different senses:

First: At 42°N, during the eight days between March 11 and March 19, the time of sunrise gets 13 minutes earlier. We thus would say that on March 15 at 42°N, sunrise is changing at –1.63 minutes/day.

Second: On the other hand, on March 15 we see that the time of sunrise varies by 1 minute as we go from 40°N to 44°N. We would thus say that at 42°N the rate of change of sunrise as the latitude varies is approximately 1 minute/4° = +.25 minutes/degree of latitude.

Two quite different rates are at work here, one with respect to time, the other with respect to latitude.

A function of several variables has several rates of change

We need a notation which allows us to talk about the different rates at which a function can change, when that function depends on more than one variable. A rate of change is, of course, a derivative. But since a change in one input produces only part of the change that a function of several variables can experience, we call the rate of change with respect to any one of the inputs a **partial derivative**. If the value of z depends on the variables x and y according to the rule $z = F(x, y)$, then we denote the rate at which z is changing with respect to x when $x = a$ and $y = b$ by

$$F_x(a, b) \qquad \text{or by} \qquad \frac{\partial z}{\partial x}(a, b).$$

Partial derivatives

We call this rate the **partial derivative of F with respect to x**. Similarly, we define the partial derivative of F with respect to y to be the rate at which z is changing when y is varied. It is denoted

$$F_y(a, b) \qquad \text{or} \qquad \frac{\partial z}{\partial y}(a, b).$$

There is nothing conceptually new involved here; to calculate either of these partial derivatives you simply hold one variable constant and go through the same limiting process as before for the input variable of interest. Note that, to call attention to the fact that there is more than one input variable present, we write

$$\frac{\partial z}{\partial x} \qquad \text{rather than} \qquad \frac{dz}{dx},$$

as we did when x was the only input variable.

To calculate the partial derivative of F with respect to x at the point (a, b), we can use

$$F_x(a, b) = \frac{\partial z}{\partial x}(a, b) = \lim_{\Delta x \to 0} \frac{F(a + \Delta x, b) - F(a, b)}{\Delta x}.$$

Similarly,

$$F_y(a, b) = \frac{\partial z}{\partial y}(a, b) = \lim_{\Delta y \to 0} \frac{F(a, b + \Delta y) - F(a, b)}{\Delta y}.$$

By using this notation for partial derivatives, we can cast some of our earlier statements about the sunrise function $T = T(d, \lambda)$ in the following form:

$$T_d(74, 42°\text{N}) \approx -1.63 \text{ minutes per day;}$$
$$T_\lambda(74, 42°\text{N}) \approx +.25 \text{ minutes per degree.}$$

Partial Derivatives as Multipliers

For any given date d and latitude λ we can write down two microscope equations for the sunrise function $T(d, \lambda)$. One describes how the time of sunrise responds to changes in the date, the other to changes in the latitude. Let's consider variations in the time of sunrise in the vicinity of March 15 and 42°N.

The partial derivative $T_d(74, 42°\text{N})$ of T with respect to d is the multiplier in the first of these microscope equations:

The microscope equation for dates

$$\Delta T \approx T_d(74, 42°\text{N}) \cdot \Delta d.$$

For example, from March 15 to March 17 ($\Delta d = 2$ days), we would expect the time of sunrise to change by

$$\Delta T \approx -1.63 \, \frac{\text{min}}{\text{day}} \times 2 \text{ days} = -3.3 \text{ minutes}.$$

Thus, we would expect the time of sunrise on March 17 at 42°N to be approximately

$$T(76, 42°\text{N}) \approx 6{:}09.7.$$

The partial derivative $T_\lambda(74, 42°\text{N})$ of T with respect to λ is the multiplier in the second microscope equation:

The microscope equation for latitudes

$$\Delta T \approx T_\lambda(74, 42°\text{N}) \cdot \Delta\lambda.$$

If, say, we moved 1° north, to 43°N, we would expect the time of sunrise on March 7 to change by

$$\Delta T \approx .25 \, \frac{\text{min}}{\text{deg}} \times 1 \text{ degree} = .25 \text{ minute}.$$

The time of sunrise on March 15 at 43°N would therefore be

$$T(74, 43°\text{N}) \approx 6{:}13.25.$$

We have seen what happens to the time of sunrise from March 15 to March 17 if we stay at 42°N, and we have seen what happens to the time on March 15 if we move from 42°N to 43°N. Can we put these two pieces of information together? That is, can we determine the time of sunrise on March 17 at 43°N? This involves changing *both* the date and the latitude.

The total change

To determine the total change we shall just combine the two changes ΔT we have already calculated. Making the date two days later moves the time of sunrise 3.3 minutes *earlier*, and travelling one degree north makes the time of sunrise .25 minutes *later*, so the net effect would be to change the time of sunrise by

$$\Delta T \approx -3.3 \text{ min} + .25 \text{ min} \approx -3 \text{ minutes}.$$

This puts the time of sunrise at $T(76, 43°\text{N}) \approx$ 6:10.

We can formulate this idea more generally in the following way: partial derivatives are not only multipliers for gauging the separate effects that changes in each input have on the output, but they also serve as multipliers for gauging the *cumulative* effect that changes in all inputs have on the output. In general, if $z = F(x, y)$ is a function of two variables, then near the point (a, b), the combined change in z caused by small changes in x and y can be stated by the *full* microscope equation:

The full microscope equation:

$$\boldsymbol{\Delta z \approx F_x(a, b) \cdot \Delta x + F_y(a, b) \cdot \Delta y}$$

As was the case for the functions of one variable, there is an important class of functions for which we may write "=" instead of "≈" in this relation, the linear functions. The most general form of a **linear function of two variables** is $z = F(x, y) = mx + ny + c$, for constants m, n, and c.

In the exercises you will have an opportunity to verify that for a linear function $z = F(x, y) = mx+ny+c$ and for all (a, b), we know that $F_x(a, b) = m$ and $F_y(a, b) = n$, and the full microscope equation $\Delta z = F_x(a, b) \cdot \Delta x + F_y(a, b) \cdot \Delta y$ is true for all values of Δx and Δy.

Formulas for Partial Derivatives

To find a partial derivative, treat the other variable as a constant

No new rules are needed to find the formulas for the partial derivatives of a function of two variables that is given by a formula. To find the partial derivative with respect to one of the variables, just treat the other variable as a constant and follow the rules for functions of a single variable. (The basic rules are described in section 5 and the chain rule in section 6.) We give two examples to illustrate the method.

Example 1. For $z = F(x, y) = 3x^2y + 5y^2\sqrt{x}$, we have

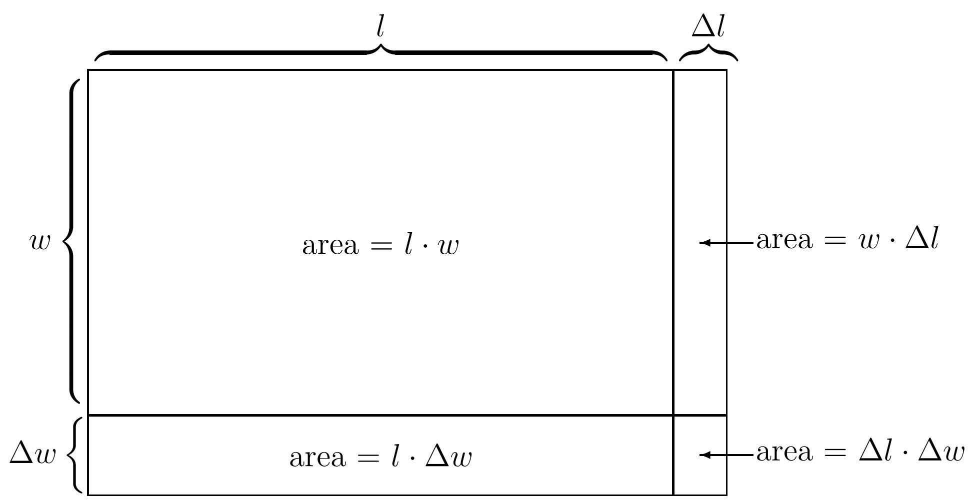
$$F_x(x, y) = 3y(2x) + 5y^2\frac{1}{2\sqrt{x}} = 6xy + \frac{5y^2}{2\sqrt{x}}$$
$$F_y(x, y) = 3x^2 + 10y\sqrt{x}$$

Example 2. For $w = G(u, v) = 3u^5 \sin v - \cos v + u$, we have

$$\frac{\partial w}{\partial u} = 15u^4 \sin v + 1$$
$$\frac{\partial w}{\partial v} = 3u^5 \cos v + \sin v$$

The formulas for derivatives and the combined multiplier effect of partial derivatives allow us to determine the effect of changes in length and in width on the area of a rectangle. The area A of the rectangle is a simple function of its dimensions l and w, $A = F(l, w) = lw$. The partial derivatives of the area are then

$$F_l(l, w) = w \qquad \text{and} \qquad F_w(l, w) = l.$$



Changes Δl and Δw in the dimensions produce a change

$$\Delta A \approx w \cdot \Delta l + l \cdot \Delta w$$

The full microscope equation for rectangular area

in the area. The picture below shows that the *exact* value of ΔA includes an additional term—namely $\Delta l \cdot \Delta w$—that is not in the approximation $w \cdot \Delta l + l \cdot \Delta w$. The difference, $\Delta l \cdot \Delta w$, is *very* small when the changes Δl and Δw are small. In chapter 5 we will have a further look at the nature of this approximation.

Exercises

1. Use differentiation formulas to find the partial derivatives of the following functions.

a) x^2y.

b) $\sqrt{x+y}$.

c) $x^2y + 5x^3 - \sqrt{x+y}$.

d) 10^{xy}.

e) $\dfrac{y}{x}$.

f) $\sin\dfrac{y}{x}$.

g) $17\dfrac{x^2}{y^3} - x^2\sin y + \pi$.

h) $\dfrac{uv}{5} + \dfrac{5}{uv}$.

i) $2\sqrt{x}\,\sqrt[3]{y} - 7\cos x$.

j) $x\tan y$.

2. On March 7 in the Northern Hemisphere, the farther south you are the earlier the sun rises. The sun rises at 6:25 on this date at 40°N. If we had been far enough south, we could have experienced a 6:25 sunrise on March 5. Near what latitude did this happen?

3. The volume V of a given quantity of gas is a function of the temperature T (in degrees Kelvin) and the pressure P. In a so-called ideal gas the functional relationship between volume and pressure is given by a particularly simple rule called the **ideal gas law**:

$$V(T,P) = R\frac{T}{P},$$

where R is a constant.

a) Find formulas for the partial derivatives $V_T(T,P)$ and $V_P(T,P)$

b) For a particular quantity of an ideal gas called a *mole*, the value of R can be expressed as 8.3×10^3 newton-meters per degree Kelvin. (The *newton* is the unit of force in the meter-kilogram-second (or MKS) system of units. Check that the units in the ideal gas law are consistent if V is measured in cubic meters, T in degrees Kelvin, and P in newtons per square meter.

c) Suppose a mole of gas at 350 degrees Kelvin is under a pressure of 20 newtons per square meter. If the temperature of the gas increases by 10 degrees Kelvin and the volume increases by 1 cubic meter, will the pressure increase or decrease? By about how much?

4. Write the formula for a linear function $F(x, y)$ with the following properties:

$$\begin{aligned} F_x(x,y) &= .15 \qquad \text{for all } x \text{ and } y \\ F_y(x,y) &= 2.31 \qquad \text{for all } x \text{ and } y \\ F(4,1) &= 8 \end{aligned}$$

5. The purpose of this exercise is to verify the claims made in the text for the linear function $z = F(x, y) = mx + ny + c$, where m, n and c are constants.

a) Use the differentiation rules to find the partial derivatives of F.

b) Use the *definition* of the partial derivative $F_x(a, b)$ to show that $F_x(a, b) = m$ for any input (a, b). That is, show that the value of

$$\frac{F(a + \Delta x, b) - F(a, b)}{\Delta x}$$

exactly equals m, no matter what a and b are.

c) Compute the exact value of the change

$$\Delta z = F(a + \Delta x, b + \Delta y) - F(a, b)$$

corresponding to changing a by Δx and b by Δy.

6. Suppose $w = G(u, v) = \dfrac{uv}{3 + v}$.

a) Approximate the value of the partial derivative $G_u(1, 2)$ by computing $\Delta w/\Delta u$ for $\Delta u = \pm.1, \pm.01, \ldots, \pm.00001$.

b) Approximate the value of $G_v(1, 2)$ by computing $\Delta w/\Delta v$ for $\Delta v = \pm.1$, $\pm.01, \ldots, \pm.00001$.

c) Write the full microscope equation for $G(u, v)$ at $(u, v) = (1, 2)$.

d) Use the full microscope equation to approximate $G(.8, 2.1)$. How close is your approximation to the true value of $G(.8, 2.1)$?

7. a) A rectangular piece of land has been measured to be 51 feet by 2034 feet. What is its area?

b) The narrow dimension has been measured with an accuracy of 4 inches, but the long dimension is accurate only to 10 feet. What is the error, or uncertainty, in the calculated area? What is the percentage error?

8. Suppose $z = f(x, y)$ and

$$f(3,12) = 240, \qquad f_x(3,12) = 7, \qquad f_y(3,12) = 4.$$

a) Estimate $f(4,12)$, $f(3,13)$, $f(4,13)$, $f(4,10)$.

b) When $x = 3$ and $y = 12$, how much does a 1% increase in x cause z to change? How much does a 1% increase in y cause z to change? Which has the larger effect: a 1% increase in x or a 1% increase in y?

9. Let $P(K, L)$ represent the monthly profit, in thousands of dollars, of a company that produces a product using capital whose monthly cost is K thousand dollars and labor whose monthly cost is L thousand dollars. The current levels of expense for capital and labor are $K = 23.5$ and $L = 39.0$. Suppose now that company managers have determined

$$\frac{\partial P}{\partial K}(23.5, 39.0) = -.12, \qquad \frac{\partial P}{\partial L}(23.5, 39.0) = -.20.$$

a) Estimate what happens to the monthly profit if monthly capital expenses increase to \$24,000.

b) Each typical person added to the work force increases the monthly labor expense by \$1,500. Estimate what happens to the monthly profit if one more person is added to the work force. What, therefore, is the rate of change of profit, in thousands of dollars per person? Is the rate positive or negative?

c) Suppose managers respond to increased demand for the product by adding three workers to the labor force. What does that do to monthly profit? If the managers want to keep the profit level unchanged, they could try to alter capital expenses. What change in K would leave profit unchanged after the three workers are added? (This is called a **trade-off**.)

10. A forester who wants to know the height of a tree walks 100 feet from its base, sights to the top of the tree, and finds the resulting angle to be 57 degrees.

a) What height does this give for the tree?

b) If the 100-foot measurement is certain only to 1 foot and the angle measurement is certain only to 5 degrees, what can you say about the uncertainty of the height measured in part (a)? (Note: you need to express angles in *radians* to use calculus: π radians = 180 degrees.)

c) Which would be more effective: improving the accuracy of the angle measurement, or improving the accuracy of the distance measurement? Explain.

3.8 Chapter Summary

The Main Ideas

- The functions we study with the calculus have graphs that are **locally linear**; that is, they look approximately straight when magnified under a microscope.

- The **slope of the graph** at any point is the **limit** of the slopes seen under a microscope at that point.

- The **rate of change** of a function at a point is the slope of its graph at that point, and thus is also a **limit**. Its dimensional units are (units of output)/(unit of input).

- The **derivative** of $f(x)$ at $x = a$ is name given to both the rate of change of f at a and the slope of the graph of f at $(a, f(a))$.

- The derivative of $y = f(x)$ at $x = a$ is written $\boldsymbol{f'(a)}$. The **Leibniz notation** for the derivative is $\boldsymbol{dy/dx}$.

- To calculate the derivative $f'(a)$, make **successive approximations** using $\Delta y/\Delta x$:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- The **microscope equation** $\boldsymbol{\Delta y \approx f'(a) \cdot \Delta x}$ describes the relation between x and $y = f(x)$ as seen under a microscope at $(a, f(a))$; $\boldsymbol{\Delta x}$ and $\boldsymbol{\Delta y}$ are the **microscope coordinates**.

- The microscope equation describes how the output changes in response to small changes in the input. The response is proportional, and the derivative $f'(a)$ plays the role of **multiplier**, or scaling factor.

- The microscope equation expresses the **local linearity** of a function in analytic form. The microscope equation is exact for **linear** functions.

- The microscope equation describes **error propagation** when one quantity, known only approximately, is used to calculate another.

- The **derivative function** is the rule that assigns to any x the number $f'(x)$.

- The derivative of a function gives information about the shape of the graph of the function, and conversely.

- If a function is given by a formula, its derivative also has a formula. There are formulas for the derivatives of the **basic functions**, and there are **rules** for the derivatives of combinations of basic functions.

- The **chain rule** gives the formula for the derivative of a **chain**, or **composite** of functions.

- Functions that have more than one input variable have **partial derivatives**. A partial derivative is the rate at which the output changes with respect to one variable when we hold all the others constant.

- If a multi-input function is given by a formula, its partial derivatives also have formulas that can be found using the same rules that apply to single-input functions.

- A function $z = F(x, y)$ of two variables also has a **microscope equation**:

  $$\Delta z \approx F_x(a, b) \cdot \Delta x + F_y(a, b) \cdot \Delta y.$$

  The partial derivatives are the **multipliers** in the microscope equation.

Expectations

- You should be able to approximate $f'(a)$ by zooming in on the graph of f near a and calculating the slope of the graph on an interval on which the graph appears straight.

- You should be able to approximate $f'(a)$ using a table of values of f near a.

- From the microscope equation $\Delta y \approx f'(a) \cdot \Delta x$, you should be able to estimate any one of Δx, Δy and $f'(a)$ if given the other two.

- If $y = f(x)$ and there is an error in the measured value of x, you should be able to determine the absolute and relative error in y.

- You should be able to sketch the graph of f' if you are given the graph of f.
- You should be able to use the basic differentiation rules to find the derivative of a function given by a formula that involves sums of constant multiples of x^p, $\sin x$, $\cos x$, $\tan x$, or b^x.
- You should be able to break down a complicated formula into a chain of simple pieces.
- You should be able to use the chain rule to find the derivative of a chain of functions. This could involve several independent steps.
- For $z = F(x, y)$, you should be able to approximate any one of Δz, Δx, Δy, $F_x(a, b)$ and $F_y(a, b)$, if given the other four.
- You should be able to find formulas for partial derivatives using the basic rules and the chain rule for finding formulas for derivatives.