A Mathematical Art

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This is a column about the mathematical structures that give us pleasure. Usefulness is irrelevant. Significance, depth, even truth are optional. If something appears in this column, it's because it's intriguing, or lovely, or just fun. Moreover, it is so intended.

A work of art is a CONSTRUCTION called into being by the artist who … has felt compelled to communicate [his ideas] to his fellow human beings. A research mathematician plays with ideas, which he puts together into what may vaguely be called STRUCTURES … In ways, possibly quite similar to the ways of the artist, [the mathematician] makes CONSTRUCTIONS. In this sense he is also an artist.

—Zoltan Dienes

The words of the pioneering educator, mathematician, philosopher, psychologist, and poet Zoltan Dienes (1916–2014) suggest that the focus of the column "For Our Mathematical Pleasure" might be an ART. Is that possible?

The beauty of mathematics has been celebrated for thousands of years, but it's a big step, philosophically, to go from "beautiful" to "art." We're going to take that step. Outline of the step:

1. The chief difficulty in defining the art of mathematics
2. Defining the art of mathematics
3. Deciding when a work is good
4. The appreciative public
5. Some important artists
6. A few masterpieces
7. Consulting with philosophers
8. Consulting with historians

On the advice of our legal team (and at the alarmed insistence of our editor), we include

9. Disclaimers

To end, I'll give you

10. A new (possibly) work of art, which will definitely satisfy point 2. Readers can judge whether it satisfies point 3 and whether they belong to 4.

This column may be controversial. That would be fun! Please write me at pleasingmath@gmail.com with your comments!

The Chief Difficulty

The chief difficulty in locating an art within mathematics is deciding what the objects of such an art could be. Finding beautiful specimens is not enough. A sunset, for example, may be stunning, but sunsets appear without human

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agency. An art object should involve the participation of an artist.

As a first guess for art object we might consider theorems. But theorems are necessary truths. They're a bit like sunsets. Humans may discover theorems, but it's hard to say that they create them. The theorems were already there; in fact, they existed before the big bang. Discovery doesn't seem like an act of creation. A computer can be programmed to churn out theorems—all theorems, in fact.

Proofs present a similar problem, though there is more scope in proof for creativity.[1]

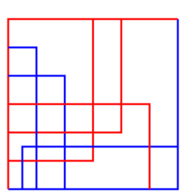
An additional difficulty with theorems and proofs is that they have to be "right"; that is, a theorem needs to be true, and proofs need to be correct. These restrictions make theorems and proofs categorically different from poems, sculptures, and symphonies. Heaven help literature if stories were required to be true!

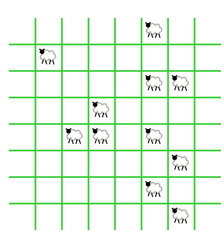
Defining the Art

Zoltan Dienes has the solution to our difficulty. The objects of our art will be mathematical structures. We only need to state carefully what we mean by such a structure. But we've already done this. In two previous columns.[2] we made a definition:

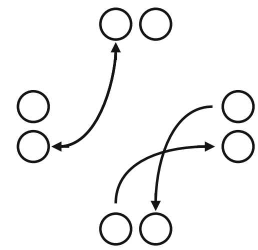
> A mathematical structure is anything that can be described completely and unambiguously.

The subject matter of mathematics consists of mathematical structures. These are exactly the objects about which we can reason logically: number systems, geometries, algebras, combinatorial structures—everything we can define precisely, from the finite to the infinite. Besides standard mathematics there are mathematical structures enjoyed by nonmathematicians, for example games and puzzles,[3]





origami, card magic, and even dance,



One could argue that the substance of the art I am describing isn't mathematics. Instead, it's the raw material of mathematics. It's what mathematicians work with. Well—perhaps it isn't mathematics. But when you read the next section, I think you will agree that the art is *mathematical.*

By the way, my introduction to Zoltan Dienes was through his book *Mathematics through the Senses, Games, Dance and Art.*[4] I was enchanted by a three-person dance there that offered primary school students a chance to innocently explore S_3, the nonabelian group of order 6.

Deciding When a Work Is Good

How can we say when a mathematical structure is successful? What is success?

If we like a poem, we'll read it again and again. Each time we sit down with it, we get more out of it, more meaning, more satisfaction. If we like a song, we'll listen to it over and over, finding more in it to appreciate. We might sing it ourselves. If we like a painting, we'll revisit it. We may acquire a reproduction and put it on a wall. We'll enjoy it at a deeper and deeper level.

That suggests an operational definition: A successful mathematical structure is one that we or others want to play with and explore. Time spent with a good structure is rewarded. We learn more about it; we plumb its depths. Tantalizing questions about the structure bubble up, questions that we enjoy wrestling with whether or not we can answer them. And we appreciate what others have discovered.

A qualitative definition is more difficult. There are many aesthetics that have been identified for mathematics over the centuries. For now, we propose just two. The first is simplicity. By this we mean *simplicity of presentation*. A mathematical structure is lovelier if it can be briefly and cleanly described.

The second is complexity. By this we mean *complexity of consequence*, depth. A mathematical structure is more

[1] For a lovely example, see Stan Wagon's award-winning paper "Fourteen Proofs of a Result about Tiling a Rectangle," *Amer. Math. Monthly* 94:7 (1987), 601–617.

[2] *Mathematical Intelligencer* 40:1 (2018) and 41:1 (2019).

[3] *Mathematical Intelligencer* 41:1 (2019) and 40:3 (2018).

[4] NFER Pub. Co., Windsor, England, 1973.

exciting, more enticing if there is always more to discover, if time spent with it is rewarded by greater and greater understanding.

I like these two, simplicity and complexity. They sound contradictory but of course they aren't. Furthermore, the combination of simplicity and complexity is *elegance*, the classic mathematical aesthetic.

The Appreciative Public

Many mathematicians, Poincaré for example, have argued that the beauty of mathematics can be accessed only by elite mathematicians.[5] Fortunately that's not the case for the mathematical art offered here.

Our mathematical public includes everyone who encounters and judges mathematical structures. Besides mathematicians, that includes all students of mathematics, anyone who enjoys puzzles, and anyone who plays games. Painters and architects are members of the mathematical public. Writers respecting the "rule of three" are members too. Anyone who appreciates the "cut and choose" method of dividing a cake is a member. Audiences laughing at logical jokes are members.[6]

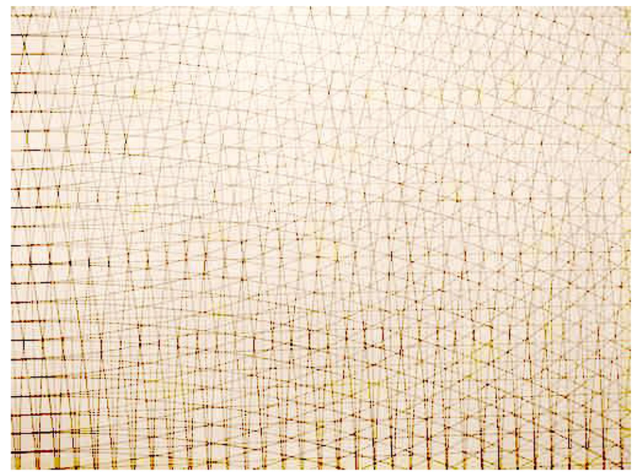
Cousin Kate: Now that you are well off, Charlie, you mustn't let them say of you, "A fool and his money are soon parted."

Cousin Charles: No, you bet I won't. I'll show them that I am an exception to the rule.

Some Mathematical Artists

In two previous columns I discussed the artists Raymond Smullyan and Sid Sackson—though I didn't call them "artists."[7] Other columns featured artists producing beautiful puzzles and engaging mathematical structures.[8] Future columns will highlight other artists—professionals and amateurs, adults and children, mathematicians and muggles.

Mathematical artists who are well known outside mathematics include Ernő Rubik, M. C. Escher, Roger Penrose, Stephen Wolfram, and George and Vi Hart. A case can be made for including the (ordinary) artist Sol LeWitt. Some of his works are not physical, they're algorithms for producing images and objects, algorithms that are complete and unambiguous. One such image graces a wall in the math department at Smith College.



Sol LeWitt, "Wall Drawing #139" (detail).

Martin Gardner deserves special mention here. His column in *Scientific American*, "Mathematical Games," featured tantalizing mathematical structures. His writings brought into being a population of like-minded individuals who love what I am calling art, who rejoice in savoring it and in creating it. Gardner always denied that he was a mathematician. But he created outstanding puzzles himself. I wonder whether he would accept the label "mathematical artist." He was a connoisseur of mathematical art, but he appreciated everyone's efforts. He was the consummate enthusiast.

A Few Masterpieces

I could fill volumes with masterpieces dating from prehistory, works of folk art such as the triangle and the square, arithmetic, prime numbers, and so on. Almost every mathematical era has contributed structures that will live forever—Euclidean geometry, algebra, the calculus, complex numbers, etc.

It is perhaps more revealing to look at recent works, especially structures that are appreciated outside the mathematics community. Here we can list as examples Rubik's cube, Conway's Life, Penrose tiles, Newcomb's paradox, flexagons, stereograms, rep-tiles, sudoku, and the games of Sprouts, Hex, and SET.

[5]Nathalie Sinclair and David Pimm, "A Historical Gaze at the Mathematical Aesthetic," in *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*, Nathalie Sinclair, David Pimm, and William Higginson, editors, Canadian Mathematical Society, 2006.

[6]Cartoon by Charles Dana Gibson (1867–1944).

[7]"The Entertainer" and "Treasures of Sid Sackson," *Mathematical Intelligencer* 40:2 (2018) and 41:1, (2019).

[8]"Meaning to Please," "Puzzle Ninja Ninja," and "Baseball Retrograde Analysis," in *Mathematical Intelligencer* 40:1, 3, 4 (2018).

Pleasing and elegant structures are all around us—seating arrangements, voting systems, necktie knots, pop-up books, shoe-lacings, card tricks. Some of these have been identified and explored. Undoubtedly countless other structures await discovery.

Consulting with Philosophers

I discussed the ideas here with some philosophers of art. They were generally supportive. We discussed the necessity of structures. A structure isn't a necessary truth, but some structures do seem to be necessary in a fundamental way. The natural numbers, for example, appear in all human cultures. If a structure is necessary in this sense, can the person who first describes it claim it as an artwork? The philosopher James Harold suggested that we might regard specialness—the opposite of necessity—as another mathematical aesthetic. In that sense, necessity is not a disqualification but a quality that can diminish a work's value.

I mentioned the worry that mathematical structures—like theorems and proofs—have always existed. But a philosopher pointed out that works of literature can be coded as finite strings of bits, which also can be said to have existed forever. We don't worry, however, that Jane Austen merely discovered *Pride and Prejudice* instead of writing it. So I've stopped worrying.

One philosopher wondered whether my use of the word "unambiguous" was unambiguous. I smiled, I hope knowingly.

The philosophers of art I spoke with were most accepting, if not enthusiastic. I am not totally relieved, however, because we are going through an epoch in the history of art in which it is hazardous to object to anything.

Consulting with Historians

I talked to a few historians of mathematics. It seemed to me that to truly qualify as a mathematical artist, "meaning to please" was important. That is, if you create something beautiful by accident—perhaps you were trying to solve an abstract problem or you were helping a scientist—then you weren't functioning as an artist. I asked historians of mathematics what they could say about the intentions of mathematicians hundreds of years ago. Did any of them mean to please?

They couldn't say.

Could there have been mathematicians who cared most about beauty but defended their work as a search for truth? If so, I would want to call them artists.

The historians I consulted were reluctant to classify any historical figure as definitely an artist. Indeed, the historians were more comfortable discussing philosophical questions.

Disclaimers

I don't claim that mathematics is an art. Mathematics is huge; it contains multitudes. Mathematics has attributes of dozens of fields. All I'm doing is drawing a line around a certain fragment of mathematics and calling that fragment an art. The fragment may be other things as well. I claim it's an art.

I don't say that mathematicians are artists. Some are; some aren't. For most mathematicians, probably, beauty is significant, but discovering truths is more significant.

While I do discuss aesthetics, *I don't define mathematical "beauty."* I have thoughts. (If you ask for them, I'll share.) But like any art, the public should and will decide what is good and what is wonderful. An artist offers a mathematical structure. Either it excites and attracts interest, or it doesn't. Posterity may take a long time deciding the beauty and intrigue of a work. One can be a critic, but no individual can be the ultimate arbiter.

Then there is the dichotomy of discovery/creation. This parallels the philosophical debate between Formalism and Platonism—do mathematicians create mathematics, or do they explore an existing mathematical universe. *I don't* [in this column] *choose Formalism over Platonism*.[9] I'm not sure the issue is important.

The Sol LeWitt installation at Smith consists of two sets of concentric circles. Did he invent that idea, or was it always there? It's said that Michelangelo imagined that each sculpture he created existed already in the raw marble block and that his task was simply to free it from the surrounding marble. Perhaps discovery/creation is not a dichotomy but a duality!

Finally, there is now a well-recognized field, "Mathematics and Art,"[10] which could be confused with the subject of this column. The activity in Mathematics and Art is exciting and fruitful, but it focuses on the connections between mathematics and the arts—music, literature, architecture, etc. It is mostly about mathematics inspiring art and enabling art. Both of these roles are intriguing and I enjoy contemplating and discussing them. But in both cases, mathematics plays a supporting role. It's not the art. It supports the art.

Does this remind you of the limited role women were allowed to play in the arts for thousands of years? Male artists, poets, and troubadours celebrated the beauty of women. Women inspired and enabled creative men, but not much more. This relationship changed dramatically in the twentieth century. Mathematics changed too.

Two Mathematical Structures

It was my plan at this point to give you a particular structure dreamed up by one of my students and let you decide whether it is intriguing, fun, elegant, or complex. In other words, I wanted to see whether the general public (you) liked it as much as I do.

But just this morning, I learned that the structure was discovered some years ago and has already passed the test of time!

I'll still give you that structure, but I'll add another structure, also by one of my students, which (I think) has never appeared before. *But I won't tell you which is which.*

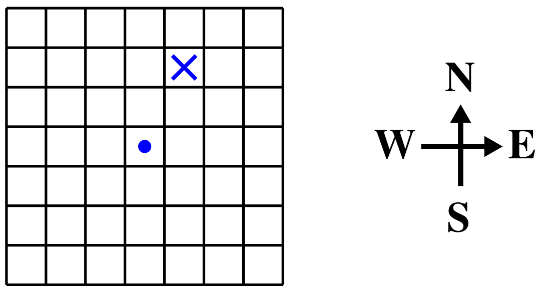
[9] Maybe I did somewhere else. In fact, I did. Maybe. That was years ago.

[10] See, for example, the *Journal of Mathematics and the Arts*, the annual *Bridges* conference on connections between mathematics and the arts, and the Wikipedia page on Mathematics and Art.

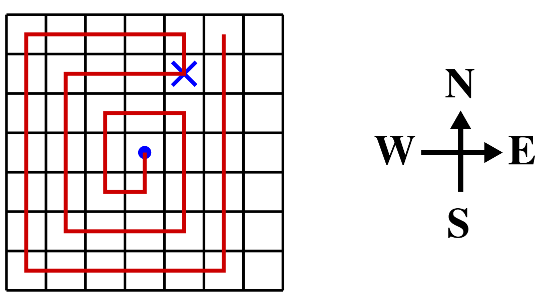
The structure with a history could be the first. And it could be the second. I'll sort them out in my next column!

A Structure by Amelia Austin

Take a square with a dot ● in it and an × somewhere else.

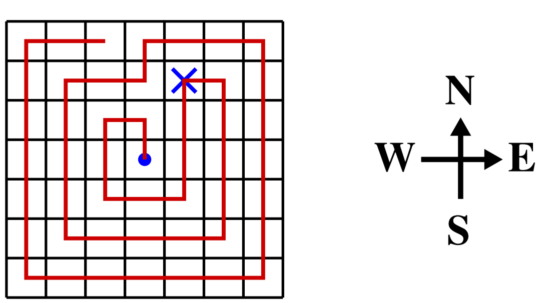


Choose a starting direction (N, S, E, or W) and a curl (clockwise or counterclockwise) proceeding from the dot. When you hit the ×, turn around and go back, curling the other way. Success means filling the entire square. Failure is getting stuck. Here's an example of getting stuck. If you choose south and clockwise, you hit the × and turn around, changing to counterclockwise,



and then you get stuck. You're stuck because it's no longer possible for you to go counterclockwise.

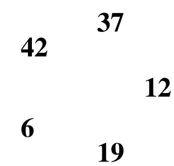
Of all the possibilities (four directions, two curls) only one combination works for this placement of × and ●, namely going north and curling counterclockwise.



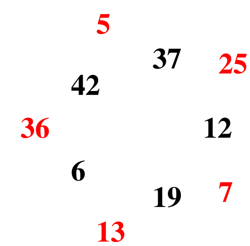
Amelia and I thought of questions. We investigated. You will either be intrigued too—or you won't!

A Structure by Halley Haruta

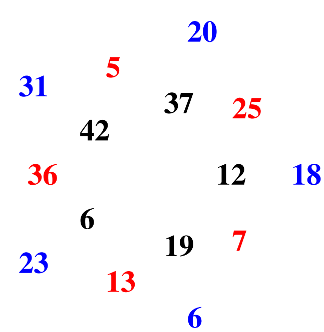
Halley was looking at circles of natural numbers. She would take a circle,



then compute the absolute differences for neighboring numbers, creating a new circle.



She kept going, creating further circles.



Halley and I had fun with this. You will too—or else you won't!

I do want to hear from you on this. If you come to a judgment (interesting, dull, cool, vapid ...) on one or both structures, please let me know at pleasingmath@gmail.com. If you are willing to share, I will post your comments on www.math.smith.edu/~jhenle/pleasingmath/, the column website.

I expect either to hear from you—or not!