Modelling Inequality with a Single Parameter[1]

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Abstract *We argue that the Lorenz curve for income is well-modelled by members of the one-parameter family of functions:*

$$\{y = (1 - (1 - r)^k)^{\frac{1}{k}}\}.$$

We justify this statement with data from the Luxembourg Income Study. The family of curves arises from a dynamic model of income growth, in which the parameter k has a direct economic interpretation.

0. Introduction

The unequal distribution of a resource is captured in all its variety by the Lorenz curve which charts, given the rank r $(0 \leq r \leq 1)$ of an entity (based on the entity's level of the resource in a given population), the proportion $L(r)$ of the resource belonging to all those of lower rank. For most of this paper, the entity will be the family and the resource will be income.

In theory, the Lorenz curve, and hence inequality in a society, is a multifaceted phenomenon. The curve is subject only to the constraints that it pass through $(0, 0)$ and $(1, 1)$ and that its derivative be non-decreasing. In practice, however, real Lorenz curves appear to follow a very distinct pattern and in nearly every case the Lorenz curve is well-modelled by a member of a one-parameter family of curves, the Lamé curves of the form:

$$\{y = (1 - (1 - r)^k)^{\frac{1}{k}}\}.$$

In section 1 we introduce our family of curves and use it to model Lorenz curves for a number of countries and years, chiefly for income data.

In section 2 we develop two economic models based on "trickle-up" theories. Both yield Lamé curves.

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[1]The authors would like to thank Peter Lambert for his generous advice and encouragement and the editor and referees for many helpful suggestions.

[2]The first author would like to dedicate his contribution to this paper to his father, the economist Peter Henle.

In section 3 we consider a number of reality checks on our model and its consequences.

In section 4 we pose a few questions.

For the most part, we will restrict our attention to inequality of income. The Luxembourg Income Study (LIS) provides excellent income data for many countries and many years. Data on the distribution of wealth are less reliable or comparable.

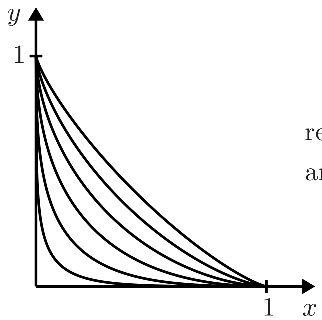
We will use r to denote the rank ($0 \le r \le 1$) of a family in terms of its income and $I(r)$ for the income of a family at rank r. We will use N for the number of families and A for the aggregrate income of all families (i.e. $A = N \int_0^1 I(x)\,dx$). The Lorenz curve, $L(r)$, is the fraction of income earned by families of rank $\le r$, that is, $L(r) = \frac{\int_0^r I(x)\,dx}{\int_0^1 I(x)\,dx} = \frac{N}{A}\int_0^r I(x)\,dx$. Alternatively, we can write: $L'(r) = \frac{I(r)}{\frac{A}{N}}$, that is, the slope of the Lorenz curve at every rank is equal to the ratio of the income of a family at that rank to the mean income for all families.[3] For background on this and on inequality in general, see Lambert (2001).

1. Modelling the Lorenz Curve

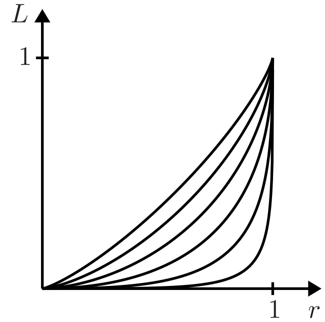
The problem of modelling the Lorenz curve has a history going back at least 40 years. Early models range from simple, $L = 1 - (1-r)^k$ (Quandt XX), $L = re^{-k(1-r)}$ (Kakwani and Podder, 1973), $L = \frac{e^{kr}-1}{e^k-1}$ (Chotikapanich, 1993) to quite elaborate $(1-(1-r)^j)^k$ (Rasche, Gaffney, Koo, and Obst [RGKO]), $r^l(1-(1-r)^j)^k$ (Sarabia, Castillo, and Slottje, 1999), $\frac{1}{\sqrt{2}}(L+r) = k\left(\frac{1}{\sqrt{2}}(L-r)\right)^j\left(\sqrt{2}-\frac{1}{\sqrt{2}}(L-r)\right)^l$ (Kakwani and Podder, 1976). Our curves are a special case of [RGKO], though we arrived at them from a different direction, as a special case, $x^k + y^k = 1$, of the Lamé curves, $\left(\frac{x}{a}\right)^k + \left(\frac{y}{b}\right)^k = 1$. These were studied by the Danish engineer and designer Piet Hein and have been called, when $k > 1$, "superellipses."

Our family of functions results from taking $a = b = 1$ and $k < 1$,

[3] In particular, $L'(.5)$ is equal to the ratio of median income to mean income.



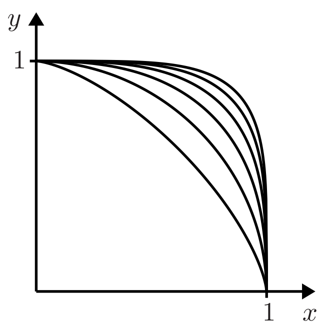
replacing x with $1 - r$
and replacing y with L,



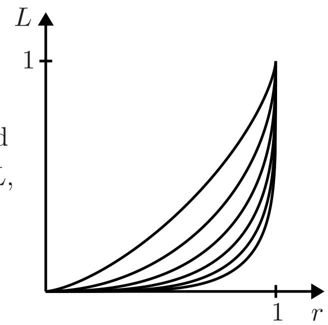
giving us

$$L(r) = (1 - (1 - r)^k)^{1/k}.$$

A second family can be formed by taking $a = b = 1$ and $k > 1$,



replacing x with r and
replacing y with $1 - L$,



giving us $L(r) = 1 - (1 - r^k)^{1/k}$.

This second family has basically the same properties as the first. For this reason and since it appears to fit the data no better and no worse, we will concentrate on the first family.

The curves satisfy the four conditions for representing a Lorenz curve as stated in Kakwani and Podder (1973): (a) $L(0) = 0$, (b) $L(1) = 1$, (c) $L(r) \leq r$, and (d) $L', L'' > 0$. It is clear that $L = (1 - (1 - r^k))^{1/k}$, $0 < k < 1$, satisfies (a) and (b). For (d), if we differentiate $L^k + (1 - r)^k = 1$ implicitly, we find that $L' = \left(\frac{1-r}{L}\right)^{k-1}$ is positive, since $r < 1$. Differentiating this yields $L'' = (1 - k)\left(\frac{1-r}{L}\right)^{k-2} \cdot \frac{L+(1-r)L'}{L^2}$, also positive, since $k < 1$. Condition (c) follows from the other three.

The family $L(r) = (1 - (1 - r)^k)^{1/k}$ is easily seen to be symmetric about the line $L = 1 - r$, that is, whenever a point (a, b) is on a curve, so is the point $(1 - b, 1 - a)$. Members of the family do not intersect (see section 3A).

The Gini coefficient of $L = (1 - (1 - r)^k)^{1/k}$ is not easily computable. Following [RGKO], it is $1 - \frac{2}{k}B(\frac{1}{k}, \frac{1}{k} + 1)$, where B is the Beta distribution.

An expression for the income distribution, since it is a multiple of the reciprocal of the second derivative of L, is relatively easy to compute. We have

$$\left(\frac{d^2L}{dr^2}\right)^{-1} = \frac{1}{1-k}(1-r)^{2-k}(1-(1-r)^k)^{2-1/k}.$$

As mentioned earlier, our family is a special case of the two parameter family proposed in [RGKO], $(1-(1-r)^j)^k$. Necessarily, because of the additional parameter, they achieve a better fit. McDonald (1984) catalogued a hierarchy of probability models (ranging from one to four parameters) for the size distribution of income. We are struck, however, by how well real Lorenz curves can be modelled without additional degrees of freedom. In addition, for many situations where data are limited (i.e. estimates are available only at the decile level), it is less clear that the additional flexibility introduced with more than one parameter is worth increased complexity of interpretation.

We tested our family of Lorenz curves, $L(r) = (1-(1-r)^k)^{1/k}$, on 89 sets of data from LIS. Each set consisted of decile data for a country and year, specifically, Austria (4 years), Australia (4 years), Belgium (4 years), Canada (8 years), Denmark (4 years), Finland (4 years), France (4 years), Germany (8 years), Ireland (4 years), Israel (4 Years), Italy (3 years), Mexico (6 years), Netherlands (4 years), Norway (4 years), Taiwan (4 years), Sweden (7 years), United Kingdom (8 years), and the United States (5 years). We also took two sets of data for the United States from Ryu and Slottje (1996) which in addition to decile points included values at $r = .91, .92, \ldots .99$. The years considered ranged from 1967 to 2000; median: 1991. The nonlinear least squares regression function `nl` in *Stata* version 9.1 was used for estimation.

We compared three measures of goodness of fit for each of the deciles within each observation (country/year). These included the root mean square error (square root of the average square of the residual), the mean absolute deviation of the observed and predicted value within a country/year as well as the maximum absolute deviation within a country/year.

The results for the 91 observations are impressive:

Variable	Mean	Std. Dev.	Min.	Max.
root mean square error (MSE)	.0043318	.0032447	.0004838	.0200169
mean absolute deviation	.003455	.0026484	.0003716	.0153033
maximum absolute deviation	.0074414	.0060083	.0007958	.0446752

The average root mean square error for the models overall was 0.0043. The maximum absolute deviation of the predicted value from any observed value was 0.045, and the largest MSE for any country/year combination was 0.020 (Italy in 1991 for both).[4] The largest MSE for any other country/year combination

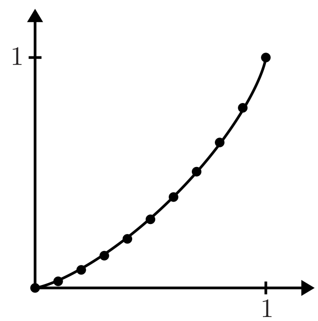
[4] The numbers for Italy in 1991 are suspect. The first decile is negative (-.01), the only

(not including US Sarabia) was 0.010 (US 1991) with corresponding maximum absolute deviation of 0.018.

35.2% of the models fit for each country/year combinations yielded a max absolute deviation of less than 0.005; 80.2% were always within 0.01 of the observed value.

The proportion of variance accounted for by the single parameter model was quite high (all R^2 values ≥ 0.998). While the addition of a second parameter may lead to a statistically significant better fit, it is less clear whether this is of practical significance.

The figure below shows a typical example, LIS data for Canada in 1997, together with the graph of $y = (1 - (1 - r)^{.752})^{1/.752}$.

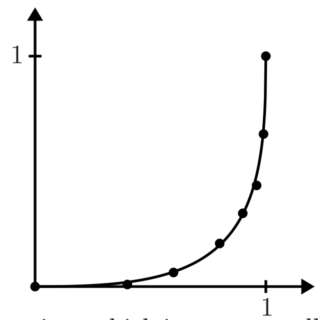


r	actual L	predicted L
0.1	0.028702916	0.032586514
0.2	0.07829257	0.083439991
0.3	0.139985671	0.145979888
0.4	0.212949199	0.218817948
0.5	0.297715202	0.301811658
0.6	0.394426981	0.395656756
0.7	0.504552741	0.502055849
0.8	0.631282881	0.624556416
0.9	0.781668058	0.771777539

The results from the one parameter model explain 99.99% of the variability, with mean absolute deviation of 0.0046 and maximal deviation of 0.0098. A plot of the residuals indicates that while these deviations are of relatively small magnitude, the primary lack of fit is due to the symmetry assumption of the one-parameter model. The two parameter model of Sarabia and colleagues provides an even better fit (mean absolute deviation of 0.0003 and maximal deviation of 0.0008) but at the cost of potentially overfitting the data, and with less readily interpretable parameters.

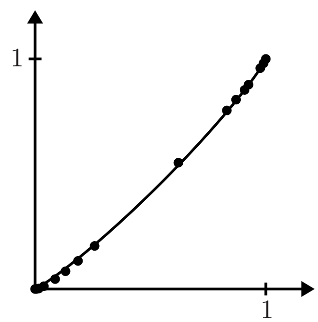
Wealth, which is more unequally distributed than income, was also well modelled by members of this family of curves. Shown below is wealth data for the United States in 1983 from Wolfe (2000) with $y = (1 - (1 - r)^{.417})^{1/.417}$.

example in all the data from LIS. In addition, the first decile was positive five years earlier (+.03) and positive again four years later (+.02).



r	actual L	predicted L
0.4	0.009	0.019077141
0.6	0.061	0.063880125
0.8	0.187	0.179748417
0.9	0.318	0.314327595
0.96	0.439	0.483775604
0.99	0.662	0.683838571

Education, which is more equally distributed than income, again fits the pattern. The figure below shows data for years of educational attainment among U.S. citizens 15 years and older, modelled by $y = (1-(1-r)^{.8862})^{1/.8862}$.



r	actual L	predicted L
0.016	0.002	0.008217649
0.038	0.012	0.021841201
0.087	0.043	0.055800117
0.132	0.077	0.089597842
0.186	0.122	0.13245114
0.258	0.187	0.192658887
0.621	0.549	0.537404018
0.831	0.776	0.769843507
0.871	0.823	0.818250894
0.908	0.865	0.864894806
0.925	0.888	0.887113172
0.976	0.96	0.958697528
0.99	0.981	0.980963166

The data, from the U.S. Census Bureau (1998), was broken down into enough categories to yield 16 points on the Lorenz curve (see Appendix A).

2. Modelling the Redistribution of Wealth

One justification for our family of Lorenz curves, $L(r) = (1-(1-r)^k)^{1/k}$, is its success in matching real Lorenz curves. We have a second justification. The family of curves is the solution to a simple dynamic model of income growth which we present here.

We start by viewing I and L as functions of two variables, rank and time. We imagine income rising or falling for each family, which in turn affects the Lorenz curve.

We also adopt a sort of "trickle-up" theory. This theory posits that families earn money off families of lower rank. The higher the rank of a family, the faster its income will grow. In other words, we assume that $\frac{\partial I}{\partial t}$ is related to $\frac{\frac{A}{N}L}{N(1-r)} = \frac{AL}{N^2(1-r)}$. $\frac{A}{N}L$ is the aggregate income of families of lower rank. $N(1-r)$ is the number of families of higher rank (a family at rank r must share development rights on poorer families with all richer families).

How is $\frac{\partial I}{\partial t}$ related to $\frac{\frac{A}{N}L}{N(1-r)} = \frac{AL}{N^2(1-r)}$? It seems reasonable to assume that $\frac{AL}{N^2(1-r)}$, and $\frac{\partial I}{\partial t}$ are simultaneously zero or simultaneously non-zero. Thus we can allow for considerable possibilities by assuming that their logarithms satisfy a linear equation. This leads to

$$\log\left(\frac{\partial I}{\partial t}\right) = B\log\left(\frac{AL}{N^2(1-r)}\right) + C,$$

or,

$$\frac{\partial I}{\partial t} = e^C\left(\frac{AL}{N^2(1-r)}\right)^B,$$

for some constants B and C.

Now, for a small interval of time Δt, we have $\Delta I = e^C\left(\frac{AL}{N^2(1-r)}\right)^B \Delta t$. For a fixed rank r, we have:

$$L + \Delta L = \frac{\int_0^r (I+\Delta I)\,dx}{\int_0^1 (I+\Delta I)\,dx} = \frac{\frac{A}{N}L + \int_0^r e^C\left(\frac{AL}{N^2(1-x)}\right)^B \Delta t\,dx}{\frac{A}{N} + \int_0^1 e^C\left(\frac{AL}{N^2(1-x)}\right)^B \Delta t\,dx}.$$

We are interested in shape of L in the steady-state, that is, when $\Delta L = 0$. This reduces the equation to:

$$L\int_0^1\left(\frac{L}{1-x}\right)^B dx = \int_0^r\left(\frac{L}{1-x}\right)^B dx.$$

The integral, $\int_0^1\left(\frac{L}{1-x}\right)^B dx$ is a constant; we will call it H. Taking the derivative of both sides with respect to r, we have:

$$H\frac{dL}{dr} = \left(\frac{L}{1-r}\right)^B$$

We can solve this equation by separation of variables:

$$\begin{aligned} H\int L^{-B}dL &= \int(1-r)^{-B}dr \\ \frac{H}{1-B}L^{1-B} &= \frac{-1}{1-B}(1-r)^{1-B} + F \\ HL^{1-B} &= -(1-r)^{1-B} + F(1-B). \end{aligned}$$

If we relabel $k = 1-B$, this simplifies to

$$HL^k + (1-r)^k = Fk.$$

In practice, $k > 0$. Substituting the points $r = 1$, $L = 1$ and $r = 0$, $L = 0$, gives us that $H = 1$ and $F = \frac{1}{k}$ and we are left with

$$L^k + (1-r)^k = 1, \quad \text{or,} \quad L = (1-(1-r)^k)^{1/k}.$$

We can attempt a corresponding "trickle-down" theory by assuming that $\frac{\partial I}{\partial t}$ depends on $\frac{\frac{A}{N}(1-L)}{Nr}$—a family at rank r developing, with those of lower rank (Nr), the wealth of those of higher rank ($\frac{A}{N}(1-L)$). From

$$\frac{\partial I}{\partial t} = e^C \left(\frac{A(1-L)}{N^2 r} \right)^B$$

we derive the second family of Lamé curves mentioned earlier, $L = 1-(1-r^k)^{1/k}$. But in this case, the constant $k = 1-B$ is greater than 1, meaning B is negative. In other words, we are left with another trickle-up theory, which one might describe as a dollar in the hands of someone at rank r sharing development rights on families of lower rank with all the dollars in the hands of those of higher rank.

These trickle-up theories suggest two additional theories, one in which dollars develop dollars (with $\frac{\partial I}{\partial t}$ proportional to $\frac{1-L}{L}$) and one in which people develop people (with $\frac{\partial I}{\partial t}$ proportional to $\frac{1-r}{r}$). Both of these result in one-parameter families, but the families lack closed-form expressions because the differential equations can't be solved analytically. The MSEs associated with these approaches are of the same order of magnitude as those associated with the other approaches.

A discussion of these alternative trickle-up theories is beyond the scope of this paper. Initial investigation, however, has persuaded the authors that it would be difficult to argue that any one is signficantly better than the others.

3. Checks and Balances

The success of the Lamé curves suggests that there is something fundamentally one-dimensional about inequality. That is a radical hypothesis that should be treated with caution. We explore the hypothesis and its ramifications here.

A. Lamé curves, as solutions to a differential equation, do not cross. But Lorenz curves do cross. Kakwani (1984) reports that in a collection of Lorenz curves 21% of the pairs intersected. Does this falsify our hypothesis?

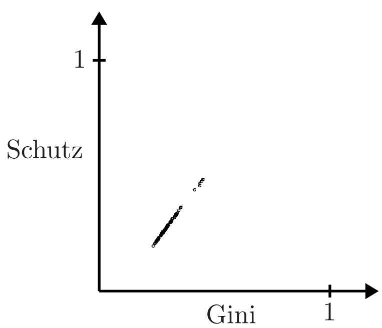
We don't believe it does. Consider what we might find if the Lorenz curve for a particular time and place were computed from two independently collected data sets. The curves would follow the same basic arc but would vary up and down. The two would almost certainly cross several times. For countries whose Lorenz curves are close, it doesn't seem surprising that they would cross.

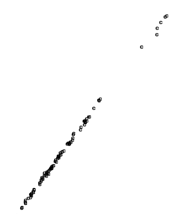
B. We have just defended the hypothesis by appealing to possible errors or random variation in the data. But the data are also the basis for our argument. Is that a difficulty?

The LIS data, we understand, is the gold-standard for income data, yet we did experience some difficulties with it. The program supplied for computing deciles, for example, had a bug. Even after dealing with that, we found at least one set of numbers that raised suspicions.[5] But unless the data have systemic biases, it seems a reasonable source on which to base our models. Note that our confidence in the data does have limits. We have four different one-parameter families (the two presented in the Introduction and the two noted at the end of the previous section), all of which model Lorenz curves well, but we don't feel we can distinguish among them.

C. If Lorenz curves were fundamentally Lamé curves, then all monotonic measures of inequality would be equivalent in the sense that knowing one measure gives you all the others. Suppose, for example, we knew the Gini coefficient g of a Lorenz curve. Then we could find the unique Lamé curve with Gini coefficient g. From that we could compute the Schutz index. Conversely, given the Schutz index, we could recover the Gini coefficient. Further, if the computations of two measures are continuous, then plotting the measures against each other should result in a connected curve.

Indeed, that seems to be the case. Here is the plot for Gini vs. Schutz, from the LIS data. The graph on the right is a magnification.





Another measure of inequality is suggested by the trickle-up theory, the exponent B in the partial differential equation,

$$\frac{\partial I}{\partial t} = e^C \left(\frac{AL}{N^2(1-r)} \right)^B .$$

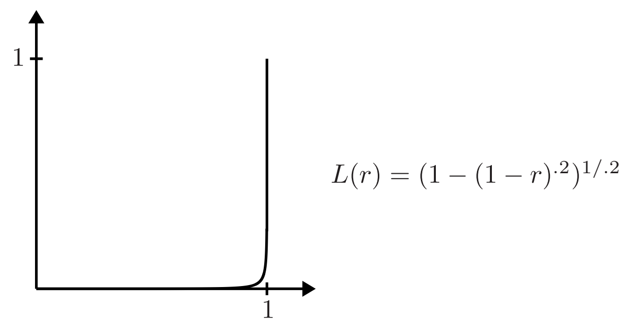
Since $B = 1 - k$, B can be determined from the best-fitting model of the Lorenz curve from the family $L(r) = \{(1 - (1 - r)^k)^{1/k}\}$. We could call this measure the "sensitivity factor" since it reflects how sensitive income growth for an individual is to the incomes of others. Perfect equality occurs when sensitivity is zero ($B = 0$, $k = 1$):

$$L(r) = 1 - (1 - r)^1)^{1/1} = r.$$

[5]Italy, 1991, as mentioned in the previous section.

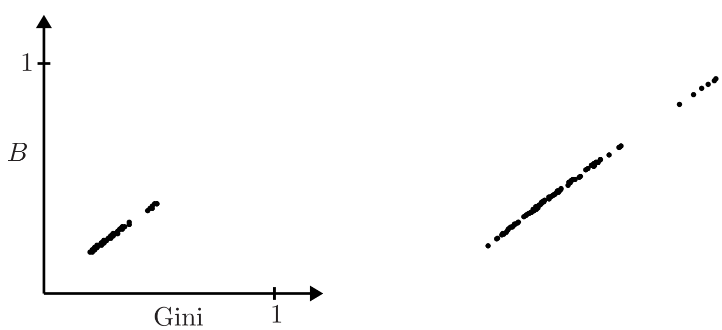
In that case, all incomes grow at the same absolute rate: $\left(\frac{\partial I}{\partial t} = e^C\right)$. At steady-state, where the Lorenz curve doesn't change, incomes can still grow, but they must all grow proportionally. The only way incomes can grow at the same absolute rate and the same proportional rate is if they are all equal.

Similarly, as k approaches 0, the Lamé curves approach absolute inequality.

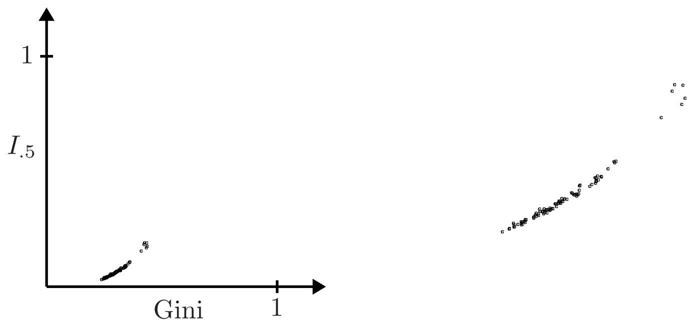


This reflects a growth rate directly proportional to what we might call the "opportunity for development," $\frac{AL}{N^2(1-r)}$.

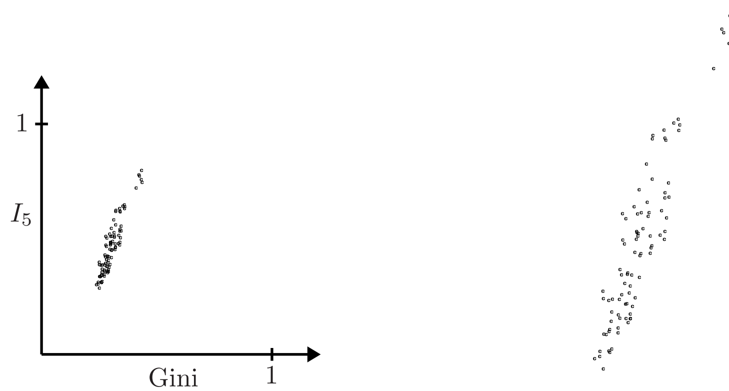
Compared to other measures, the sensitivity factor is, perhaps, less abstract and more directly meaningful. It also tracks well with the Gini coefficient (LIS data).



D. James Harvey (2005) explores the relationship between the Gini coefficient and several of the Atkinson indices I_r. His plots show large scattering which would seem to refute our hypothesis. We computed for the LIS data two Atkinson indices, one where the relationship is well-behaved in Harvey's paper, $I_{.5}$,



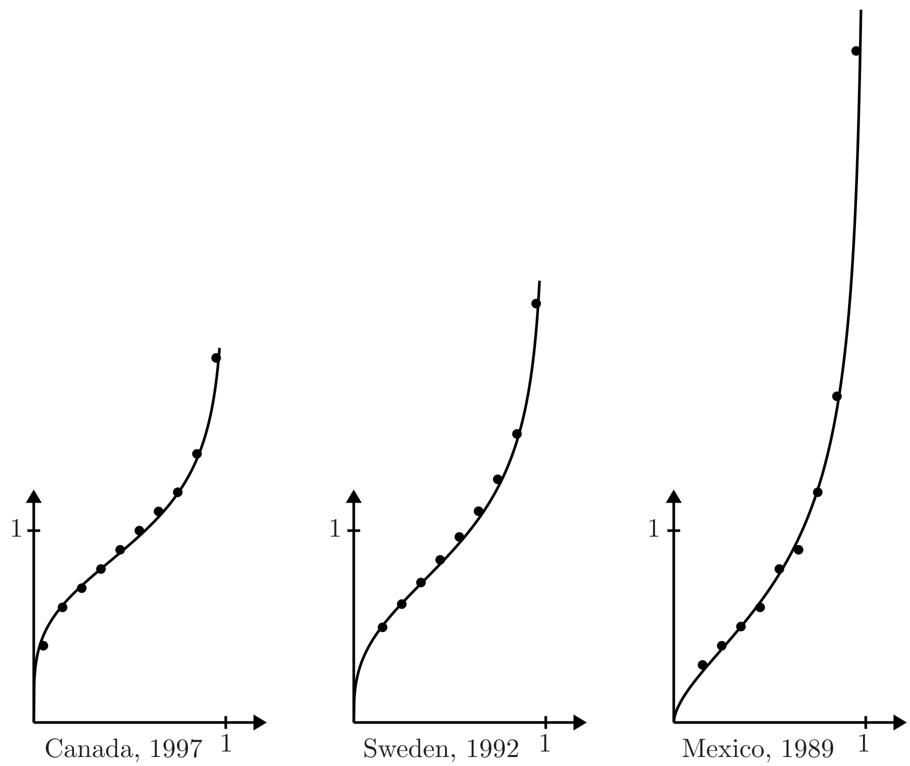
and also the one where the points are most scattered, I_5.



The picture for I_5 is not as clearly a curve as the picture for $I_{.5}$, but it's more organized and linear than the picture in Harvey (2005). The higher subscript tends to exaggerate differences at the low end of the Lorenz curve. An error, for example, of ϵ in the calculation of $L(.1)$ can change I_5 by more than 10ϵ.

E. Finally, we are modelling a curve that is confined to a small space, a curve that must go from $(0, 0)$ to $(1, 1)$ with a constantly increasing derivative. Under those circumstances, modelling closely with a carefully chosen family of curves may seem unspectacular.

We considered this and thought to test how well the derivatives of our curves matched the derivatives of the Lorenz curves (Pen's Parade, since L' is proportional to income). This is a significantly greater challenge, since the derivative is theoretically unbounded. The following graphs show the derivatives of the Lamé curves for countries with varying degrees of inequality. The points are the difference quotients formed from consecutive decile data.



The model passes this test surprisingly well.

4. Questions

We have presented evidence that Lorenz curves for income taken at different times in different countries are well-modelled by curves from a one-parameter family of functions. Of course, additional parameters produce better fits. Modern economies are subject to countless disturbances which must vary the Lorenz curve in local but significant ways.

But the closeness of the approximations produced by a single parameter implies that the distribution of income in a society is largely characterized by a single number. This raises some related questions.

1. Is there a single economic variable that drives inequality?

2. What are the ways in which the sensitivity factor can be changed through economic policy?

3. What does the success of trickle-up theories have to say about how governments should stimulate the economy?

4. Can $B = 1 - k$ be seen as a measure of the efficiency of an economy? If so,

does this suggest an explicit trade-off between efficiency and equality?

5. Inequality in the United States decreased from 1950 to 1970 (Henle, 1972) and increased from 1979 to 2000 (the Gini coefficient increased steadily from .301 to .368). Can the framework of this paper help explain these trends?

6. The relationship between the Gini coefficient and the sensitivity factor appears almost linear. Does this mean that the Gini coefficient has a concrete interpretation? That is, does the Gini coefficient tell us something definite about the relation between the rate of growth of one's income and the income of those who earn less?

7. We were not able to distinguish among the four models of income growth ($\frac{L}{1-r}$, $\frac{r}{1-L}$, $\frac{L}{1-L}$, $\frac{r}{1-r}$). Is there a way of determining which leads to the best for model for Lorenz curves?

Appendix A On the Education Data

We found a Lorenz curve for educational attainment using data from the U.S. Census Bureau. In the table below, we have noted the number of years we attached to each category.

none (years: 0)	1st-4th (years: 2.5)	5th-6th (years: 5.5)	7th-8th (years: 7.5)	9th (years: 9)	10th (years: 10)	11th (years: 11)	H.S. grad (years: 12)	Some college (years: 13)	Associate degree (years: 14)	Bachelor's degree (years: 16)	Master's degree (years: 17)	Professional Degree (years: 18)	Doctorate Degree (years: 22)
887	2091	3911	9039	8212	9795	12993	66210	38315	13998	3090	9295	2586	1869

Educational Attainment of Persons 15 Years Old and Over (all races, both sexes, in thousands)

The data gave us (with $(0,0)$ and $(1,1)$) 16 points on the Lorenz curve.

Appendix B Computational Details

For interest, we report here on the techniques we used to compute (a) the Gini coefficient and (b) the Schutz index.

(a) We computed the Gini coefficient from quintile data using a Newton-Cotes formula.

Given the value of a function f at three values, a, $a + .5(b - a)$, b, Simpson's Rule approximates the integral of f on $[a, b]$ by integrating the quadratic passing through the three points. Given the value of f at more points the Newton-Cotes formulae find more accurate approximations by integrating polynomials of higher degree. The particular formula we used (appropriate for the six points given by quintile data) approximates $\int_a^b f(x)\,dx$ by $\frac{95}{288} f(a) + \frac{125}{96} f(a + .2(b - a)) + \frac{125}{144} f(a + .4(b - a)) + \frac{125}{144} f(a + .6(b - a)) + \frac{125}{96} f(a + .8(b - a)) + \frac{95}{288} f(b)$.

(b) The Schutz index is the greatest distance between the Lorenz curve and the straight line from the origin to $(1, 1)$. The difficulty is determining this given only decile data for the Lorenz curve.

A little calculus tells us that the point where this distance is greatest is where $L'(r) = 1$. For most Lorenz curves, that comes when r is between .6 and .7. We then approximate L with the cubic passing through the points, $(.5, L(.5))$, $(.6, L(.6))$, $(.7, L(.7))$, $(.8, L(.8))$ and use this to find a such that $L'(a) = 1$. and then to evaluate $a - L(a)$ (the Schutz index).

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