Non-nonstandard Analysis:
Real Infinitesimals

J. M. Henle *

November 25, 1997

Abstract

We offer an alternative way to prove the theorems of calculus. It features the advantages of infinitesimals with the familiarity of limits. It is especially appropriate for students with some background in calculus.

1 Introduction

Infinitesimals are nicer than limits. They're conceptually simpler. They're easier to use. They're also more fun. Why don't we use them to teach calculus?

In short, it's because they're too strange. Three hundred years ago, infinitesimals were undefined and somewhat mystical. Today, they are thoroughly defined but still, unfortunately, mystical. They are nonconstructive, non-canonical and no more concrete than the infinitesimals of Newton and Leibniz. The Axiom of Choice is needed for their very existence. Nonstandard analysis has given us rigor, but not comfort.

There is a way, though, of constructing infinitesimals naturally. Ironically, the seeds can be found in any calculus book of sufficient age. At the turn of the century, it was typical of texts to define an infinitesimal as a "variable whose limit is zero" ([C]). That is our inspiration. Our infinitesimals will be sequences.

In what follows, we will be manipulating sequences of real numbers. We will treat them (mostly) as numbers. We will add them, subtract them, and put them into functions. They aren't numbers, however. Trichotomy fails, for example.

The central construction in this paper is a rediscovery, due first, probably, to D. Laugwitz. More on this later.

*I would like to thank Michael Henle, Ward Henson, Roman Kossak, Dan Velleman, and several anonymous reviewers for careful reading and helpful remarks.

Notation: We will denote a sequence $\{a_n\}_{n\in\mathbb{N}}$ by a boldface $\mathbf{a}$. We permit a_n to be undefined for finitely many n. The key idea throughout is that of "*for all but finitely many n.*" We abbreviate this: (FAB). An equation or inequality involving sequences will be interpreted as being true (FAB). For example,

$$\mathbf{a} > 2$$

simply means "$a_n > 2$ (FAB)," or, that for all but finitely many n, $a_n > 2$. We will use ordinary size letters for real numbers. Note that a real number r can be viewed as the sequence $\mathbf{a}$, with $a_n = r$ for all n.

Operations on and functions of sequences are performed in the usual way, that is, $\mathbf{a}+\mathbf{b} = \mathbf{c}$ means $a_n + b_n = c_n$ (FAB), and $\sin(\mathbf{a}) = \mathbf{b}$ means $\sin(a_n) = b_n$ (FAB).

Note that if a statement P is true (FAB), and statement Q is also true (FAB), then "P and Q" is true (FAB). This fact allows us to do algebra on sequences. Suppose, for example, we have

$$\mathbf{a} + 4 = 3\mathbf{b}$$
and
$$\mathbf{c} < 45.$$

Then

$$a_n + 4 = 3b_n \text{ (FAB)}$$
and
$$c_n < 45 \text{ (FAB)}.$$

Since the two are true together (FAB), we can add them to get

$$a_n + 4 + c_n < 3b_n + 45 \text{ (FAB)}.$$

In other words,

$$\mathbf{a} + 4 + \mathbf{c} < 3\mathbf{b} + 45.$$

2 Infinitely Small and Infinitely Close

Definition A sequence $\mathbf{a}$ is *infinitely small* if $|\mathbf{a}| < d$ for all positive real numbers d. For infinitely small $\mathbf{a}$, we write $\mathbf{a} \approx 0$. If $|\mathbf{a}| < r$ for some real r, we say $\mathbf{a}$ is *finite* or *bounded.* A sequence $\mathbf{a}$ is *infinitesimal* if $\mathbf{a} \neq 0$ and $\mathbf{a} \approx 0$.

Proposition 2.1 *Suppose* $\mathbf{a}, \mathbf{b} \approx 0$. *Then*
(1) $\mathbf{a} + \mathbf{b} \approx 0$.
(2) $\mathbf{a} - \mathbf{b} \approx 0$.
(3) If $\mathbf{c}$ *is finite, then* $\mathbf{ac} \approx 0$.
(4) If $|\mathbf{c}| < |\mathbf{a}|$ *then* $\mathbf{c} \approx 0$.

Proof: Given any positive d, we know that $|\mathbf{a}| < \frac{d}{2}$ and $|\mathbf{b}| < \frac{d}{2}$. Then $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}|+|\mathbf{b}| < d$. This proves (1). (2) is proved similarly.

For (3), since $|\mathbf{c}| < r$ for some real r, and $|\mathbf{a}| < \frac{d}{r}$ for all positive d, we have $|\mathbf{ac}| < d$.

For (4), observe that since $|\mathbf{c}| < |\mathbf{a}| < d$, $|\mathbf{c}| < d$. $\blacksquare_{\text{Prop. 2.1}}$

Note that reals are finite so that any result concerning finite sequences applies to reals too. Part (3) of Proposition 2.1, for example, tells us that if $\mathbf{a} \approx 0$, then $r\mathbf{a} \approx 0$ for any real r.

Definition For sequences $\mathbf{a}$ and $\mathbf{b}$, we say that $\mathbf{a}$ and $\mathbf{b}$ are *infinitely close*, or $\mathbf{a} \approx \mathbf{b}$ iff $\mathbf{a} - \mathbf{b} \approx 0$.

Proposition 2.2 *If* $\mathbf{a} \approx r$ *and* $\mathbf{b} \approx s$, *then*
(1) $\mathbf{a} + \mathbf{b} \approx r + s$.
(2) $\mathbf{a} - \mathbf{b} \approx r - s$.
(3) $\mathbf{ab} \approx rs$.
(4) If $\mathbf{a} \leq s$, *then* $r \leq s$.
(5) $\mathbf{a} \div \mathbf{b} \approx \frac{r}{s}$, *if* $s \neq 0$.

Proof: (1) and (2) follow easily from Proposition 2.1.

For (3) note that $(\mathbf{a} - r)s$, $(\mathbf{b} - s)r$, and $(\mathbf{a} - r)(\mathbf{b} - s)$ are all ≈ 0. Adding these gives us $\mathbf{ab} - rs \approx 0$.

For (4), if $r > s$, then we would have both $a_n \leq s$ (FAB) and $|a_n - r| < \frac{r-s}{2}$ (FAB), which is impossible.

In view of (3), we need only show $\frac{1}{\mathbf{b}} \approx \frac{1}{s}$ to establish (5). From $s \approx \mathbf{b}$, we get $\left|\frac{s}{2}\right| \approx \left|\frac{\mathbf{b}}{2}\right| < |\mathbf{b}|$, so by (4), $\left|\frac{s}{2}\right| \leq |\mathbf{b}|$. Then $\left|\frac{1}{\mathbf{b}} - \frac{1}{s}\right| = \left|\frac{s-\mathbf{b}}{\mathbf{b}s}\right| \leq \frac{1}{|s/2|} \cdot \frac{1}{|s|} \cdot |s - \mathbf{b}|$. By Proposition 2.1, this is infinitesimal.

$\blacksquare_{\text{Prop. 2.2}}$

This is all we need to get started.

3 Elementary Calculus

Definition 3.1 *A function* f *is* continuous *at* $x = r$ *iff*

$$\mathbf{a} \approx r \text{ implies } f(\mathbf{a}) \approx f(r).$$

Equivalently, f *is continuous if* $\mathbf{\Delta x} \approx 0$ *implies* $f(r + \mathbf{\Delta x}) - f(r) \approx 0$.

Proposition 3.1 *The sum, difference, product, and quotient (when the divisor is nonzero) of functions continuous at* $x = r$ *are continuous at* $x = r$.

This follows easily from Proposition 2.2.

Definition 3.2 *For a function f and a real r, we say $f'(r) = d$ iff for all infinitesimal $\mathbf{\Delta x}$,*

$$\frac{f(r + \mathbf{\Delta x}) - f(r)}{\mathbf{\Delta x}} \approx d.$$

Here is what the computation looks like of the standard first example: $f(x) = x^2$.

$$\begin{aligned} \frac{(x+\mathbf{\Delta x})^2 - x^2}{\mathbf{\Delta x}} &= \frac{x^2 - 2x\mathbf{\Delta x} + \mathbf{\Delta x}^2 - x^2}{\mathbf{\Delta x}} \\ &= \frac{2x\mathbf{\Delta x} + \mathbf{\Delta x}^2}{\mathbf{\Delta x}} \\ &= 2x + \mathbf{\Delta x} \\ &\approx 2x. \end{aligned}$$

Proposition 3.2 *If f is differentiable at r, it is continuous at r.*

Proof: Just multiply:

$$\frac{f(r + \mathbf{\Delta x}) - f(r)}{\mathbf{\Delta x}} \approx d \text{ and } \mathbf{\Delta x} \approx 0$$

to get:

$$f(r + \mathbf{\Delta x}) - f(r) \approx 0.$$

$\blacksquare_{\text{Prop. 3.2}}$

The proofs of the differentiation rules are simple. The product rule is typical:

Proposition 3.3 *If the function h is the product of differentiable functions f and g, then h is differentiable and $h' = f'g + g'f$.*

Proof: Writing $\mathbf{\Delta f}$ for $f(x + \mathbf{\Delta x}) - f(x)$ and similarly for g and h, $\mathbf{\Delta h} = (f + \mathbf{\Delta f})(g + \mathbf{\Delta g}) - fg$, so

$$\begin{aligned} \frac{\mathbf{\Delta h}}{\mathbf{\Delta x}} &= \frac{(f+\mathbf{\Delta f})(g+\mathbf{\Delta g}) - fg}{\mathbf{\Delta x}} \\ &= \frac{f\mathbf{\Delta g} + g\mathbf{\Delta f} + \mathbf{\Delta f}\mathbf{\Delta g}}{\mathbf{\Delta x}} \\ &= f\frac{\mathbf{\Delta g}}{\mathbf{\Delta x}} + g\frac{\mathbf{\Delta f}}{\mathbf{\Delta x}} + \mathbf{\Delta f}\frac{\mathbf{\Delta g}}{\mathbf{\Delta x}} \\ &\approx f \cdot g' + g \cdot f' + 0 \cdot g' \\ &\approx f \cdot g' + g \cdot f'. \end{aligned}$$

$\blacksquare_{\text{Prop. 3.3}}$

The proof of the Chain Rule, omitted or banished to the appendices in virtually all texts today, is easy but we need to discuss subsequences.

Definition 3.3 $\mathbf{a}$ *is a* subsequence *of* $\mathbf{b}$ *(written* $\mathbf{a} \subset \mathbf{b}$*) if every term of* $\mathbf{a}$ *is a term of* $\mathbf{b}$, *more precisely, if there is an increasing function,* $k : \mathbb{N} \Rightarrow \mathbb{N}$ *such that* $a_n = b_{k(n)}$.

Proposition 3.4 *Given* $\mathbf{a} \subset \mathbf{b}$*, then*
(1) If $\mathbf{b}$ *is infinitesimal, then so is* $\mathbf{a}$.
(2) If $\mathbf{b} \approx r$*, then* $\mathbf{a} \approx r$.
(3) If $\mathbf{b}$ *satisfies a given equation or inequality, then so does* $\mathbf{a}$.

The proof of this is routine.

Proposition 3.5 (The Chain Rule) *If* $y = f(x)$ *and* $z = g(y)$*, then* $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$.

Proof: For $\mathbf{\Delta x}$ infinitesimal,

$$\frac{dz}{dx} \approx \frac{\mathbf{\Delta z}}{\mathbf{\Delta x}} = \frac{\mathbf{\Delta z}}{\mathbf{\Delta y}} \cdot \frac{\mathbf{\Delta y}}{\mathbf{\Delta x}} \approx \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

The only difficulty here is that $\mathbf{\Delta y} = f(x + \mathbf{\Delta x}) - f(x)$, while infinitely close to 0, may not be infinitesimal because it may equal 0 infinitely often. In that case, we can't claim that $\frac{dz}{dy} \approx \frac{\mathbf{\Delta z}}{\mathbf{\Delta y}}$. But then let $\mathbf{\Delta x}^* \subset \mathbf{\Delta x}$ be the subsequence such that the corresponding $\mathbf{\Delta y}^*$ is a sequence entirely composed of 0s. Then $\frac{dy}{dx} \approx \frac{\mathbf{\Delta y}^*}{\mathbf{\Delta x}^*} = 0$. And since $\mathbf{\Delta y}^* = 0$, the corresponding $\mathbf{\Delta z}^* = g(y + \mathbf{\Delta y}^*) - g(y)$ is also 0, so $\frac{dz}{dx} \approx \frac{\mathbf{\Delta z}^*}{\mathbf{\Delta x}^*} = 0$. Thus once again,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

$\blacksquare_{\text{Prop. 3.5}}$

For integration, we can define the integral of step functions first, then:

Definition 3.4 *For a function* f *and an interval* $[p, q]$,

$$\int_p^q f dx = r$$

iff there are "step-functions" (actually sequences of step functions) $\mathbf{l} \le f \le \mathbf{u}$ *on* $[p, q]$ *with*

$$\int_p^q \mathbf{l}\, dx \approx r \approx \int_p^q \mathbf{u}\, dx.$$

The integral, r, is unique, since if $\mathbf{l_i}$, $\mathbf{u_i}$, and r_i satisfy the conditions above for $i = 1, 2$, then

$$r_1 \approx \int_p^q \mathbf{l_1}\, dx \leq \int_p^q \mathbf{u_2}\, dx \approx r_2 \approx \int_p^q \mathbf{l_2}\, dx \leq \int_p^q \mathbf{u_1}\, dx \approx r_1.$$

The basic theorems on integrals are easily proved. The following is an example:

Proposition 3.6 *If f is integrable over $[a, b]$ and $[b, c]$, then it is integrable over $[a, c]$ and*

$$\int_a^c f dx = \int_a^b f dx + \int_b^c f dx.$$

Proof: Let $\mathbf{l_{ab}}, \mathbf{u_{ab}}, \mathbf{l_{bc}}, \mathbf{u_{bc}}$ be the step functions witnessing the integrability of f over $[a, b]$ and $[b, c]$. Define $\mathbf{l}$ and $\mathbf{u}$ on $[a, c]$ by gluing together the respective lower and upper functions. Then we certainly have

$$\mathbf{l} \leq f \leq \mathbf{u}$$

and we also have

$$\int_a^c \mathbf{l} dx = \int_a^b \mathbf{l_{ab}} dx + \int_b^c \mathbf{l_{bc}} dx \approx \int_a^b \mathbf{u_{ab}} dx + \int_b^c \mathbf{u_{bc}} dx = \int_a^c \mathbf{u} dx.$$

■Prop. 3.6

4 Analysis

The outstanding power of Robinson's nonstandard analysis is evident in the nonstandard proof of theorems such as the Intermediate Value Theorem, and the integrability of continuous functions. We can do that too, and the proofs are startlingly similar. Our tool will be the non–nonstandard equivalent of "every finite nonstandard number is infinitely close to a real." The following is effectively the Bolzano-Weierstrass theorem.

Proposition 4.1 *If $\mathbf{a}$ is finite then for some $\mathbf{c} \subset \mathbf{a}$ and some real r, $\mathbf{c} \approx r$.*

Proof: Since $\mathbf{a}$ is finite, there is some d such that $|\mathbf{a}| < d$. That means there are only a finite number of possibilities for the integer part of each a_n. One of these possibilities must occur an infinite number of times. Let k be such that for infinitely many a_n, the integer part of a_n is k. Let c_1 be the first term in $\mathbf{a}$ with integer part k.

Now of the infinitely many $\{a_n\}$ having integer part k, there are only 10 possibilities $(0, 1, 2, \ldots, 9)$ for the first digit after the decimal point. One of those possibilities must occur an infinite number of times. Let d_1 be such a digit. Let c_2 be the first term in $\mathbf{a}$ after c_1 which begins "$k.d_1$."

We continue in this way, finding d_2 such that infinitely many terms begin "$k.d_1d_2$" and choosing c_3 so that it begins "$k.d_1d_2$," etc. etc. When we are done, we have a subsequence $\mathbf{c} \subset \mathbf{a}$, and a real number, $r = k.d_1d_2d_3d_4\ldots$, and by construction,

$$\begin{aligned} |c_1 - r| &< 1, \\ |c_2 - r| &< .1 \\ |c_3 - r| &< .01 \\ &\vdots \end{aligned}$$

Thus for any $q > 0$, $|c_n - r| < q$ (FAB). That means $|\mathbf{c} - r| \approx 0$, or $\mathbf{c} \approx r$. $\blacksquare_{\text{Prop. 4.1}}$

Proposition 4.2 (The Intermediate Value Theorem) *If f is continuous on $[p, q]$ and $f(p) \le s \le f(q)$, then for some $r \in [p, q]$, $f(r) = s$.*

Proof: We begin by constructing two sequences $\mathbf{a}$ and $\mathbf{b}$ in the interval $[p, q]$. For $n = 1$, let $a_1 = p$ and $b_1 = q$. For n in general, divide the interval, $[p, q]$, by points

$$p = x_0 < x_1 < x_2 < \ldots < x_n = q,$$

equally spaced, a distance of $\frac{q-p}{n}$ apart. Since $f(x_0) \le s \le f(x_n)$, there must be two adjacent points, x_k and x_{k+1} such that $f(x_k) \le s \le f(x_{k+1})$. Let a_n be the first point, x_k, and let b_n be the second.

We certainly have $f(\mathbf{a}) \le s \le f(\mathbf{b})$. We also have $\mathbf{a} \approx \mathbf{b}$. By Proposition 4.1, there is a subsequence $\mathbf{c} \subset \mathbf{a}$ and a real r such that $\mathbf{c} \approx r$. Let $\mathbf{d} \subset \mathbf{b}$ be the corresponding subsequence of $\mathbf{b}$. By Proposition 3.4, $f(\mathbf{c}) \le s \le f(\mathbf{d})$ and $\mathbf{c} \approx \mathbf{d}$. By continuity, $f(\mathbf{c}) \approx f(r) \approx f(\mathbf{d})$. Now putting everything together,

$$f(r) \approx f(\mathbf{c}) \le s \le f(\mathbf{d}) \approx f(r).$$

By Proposition 2.2(5), $f(r) \le s \le f(r)$, so $f(r) = s$. $\blacksquare_{\text{Prop. 4.2}}$

Proposition 4.3 (The Extreme Value Theorem) *If f is continuous on $[p, q]$ then f attains a maximum on $[p, q]$.*

Proof: We define a single sequence $\mathbf{a}$ in $[p, q]$. For n, divide $[p, q]$ as in the previous proof. Let a_n be the division point x_k for which $f(x_k)$ is greatest. Choose $\mathbf{c} \subset \mathbf{a}$, r so that $\mathbf{c} \approx r$. We claim that f reaches its maximum at $x = r$. To see this, take any s in $[p, q]$. Form sequence $\mathbf{b}$ by choosing b_n for each n to be the division point x_k nearest to s. We have that $\mathbf{b} \approx s$ since $|b_n - s| < \frac{q-p}{n}$, so $|b_n - s| < w$ (FAB)for any positive w. We also have, by the construction of $\mathbf{a}$, that $f(\mathbf{b}) \le f(\mathbf{a})$.

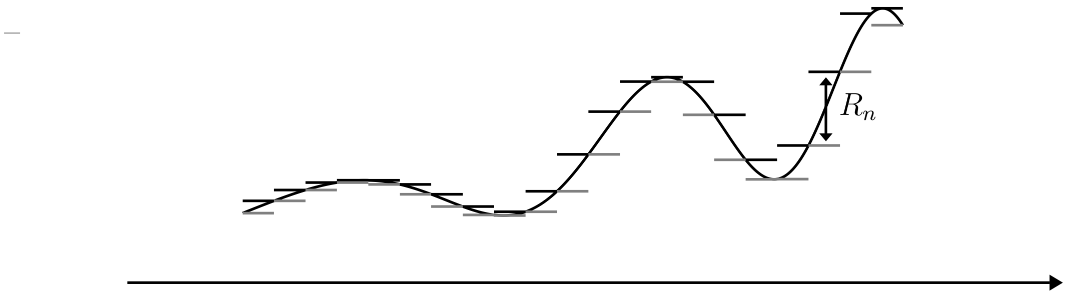
Now let $\mathbf{d}$ be the subsequence of $\mathbf{b}$ corresponding to $\mathbf{c}$. As before, $f(\mathbf{d}) \le f(\mathbf{c})$ and $\mathbf{d} \approx s$. So by continuity, $f(s) \approx f(\mathbf{d}) \le f(\mathbf{c}) \approx f(r)$. It follows that $f(s) \le f(r)$. $\blacksquare_{\text{Prop. 4.3}}$

Proposition 4.4 *If f is continuous on $[p, q]$ then f is integrable on $[p, q]$.*

Proof: For any n, let l_n be the step–function formed by partitioning $[p, q]$ into n equal subintervals and setting $l_n(x)$ on each subinterval to be the minimum value of f (guaranteed to exist by Proposition 4.3). Define u_n similarly as the maximum value of f. We have that $\mathbf{l} \leq f \leq \mathbf{u}$.

Since f is bounded, the sequences $\mathbf{L} = \int_p^q \mathbf{l}\, dx$ and $\mathbf{U} = \int_p^q \mathbf{u}\, dx$ are bounded, so there is a subsequence $\mathbf{l}^* \subset \mathbf{l}$ and real r such that the corresponding $\mathbf{L}^* = \int_p^q \mathbf{l}^* dx \approx r$. Let $\mathbf{u}^*$ be the subsequence of $\mathbf{u}$ corresponding to $\mathbf{l}^*$, and let, for each n, Δx_n be the length of the corresponding subinterval. We claim that $\int_p^q \mathbf{u}^* dx \approx r$ and so $\int_p^q \mathbf{f}\, dx = r$.

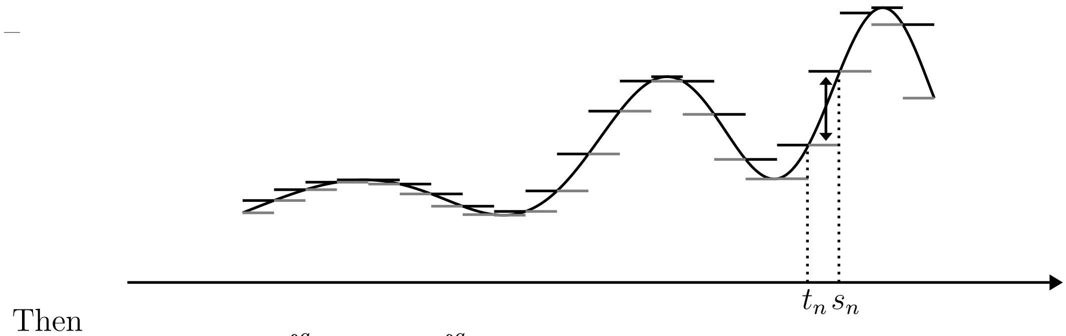
Proof of claim: For any n, we can find the greatest difference, R_n, between $l_n^*(x)$ and $u_n^*(x)$ on $[p, q]$,



and we have that

$$\int_p^q u_n^* dx - \int_p^q l_n^* dx \leq R_n(b - a).$$

The difference, R_n, can be represented as $|f(s_n) - f(t_n)|$, for some s_n, t_n with $|u_n - t_n| \leq \Delta x_n$.



Then

$$\int_p^q \mathbf{u}^* dx - \int_p^q \mathbf{l}^* dx \leq \mathbf{R}(b - a) = |f(\mathbf{s}) - f(\mathbf{t})|(b - a).$$

We would like to say that since $|\mathbf{s} - \mathbf{t}| \leq \mathbf{\Delta x} \approx 0$, so by continuity $|f(\mathbf{s}) - f(\mathbf{t})| \approx 0$, and

$$\int_p^q \mathbf{u}^* dx - \int_p^q \mathbf{l}^* dx \approx 0, \text{ and so } \int_p^q \mathbf{l}^*\, dx \approx r.$$

But continuity requires that one of $\mathbf{s}$, $\mathbf{t}$ be real. The property: $\mathbf{s} \approx \mathbf{t} \Rightarrow f(\mathbf{s}) \approx f(\mathbf{t})$ is actually equivalent to uniform continuity. We can easily work around this, however, by finding a subsequence $\mathbf{s}^{**} \subset \mathbf{s}$ and real r, $\mathbf{s}^{**} \approx r$ and using the corresponding subsequences $\mathbf{u}^{**}$, $\mathbf{l}^{**}$, and $\mathbf{t}^{**} \approx r \approx \mathbf{s}^{**}$, to finish the proof. $\blacksquare_{\text{Prop. 4.4}}$

5 How Did We Do It?

How did we avoid the Axiom of Choice? We simply asked less of our sequences than one asks of numbers. They aren't totally ordered, for example. The sequences: $0, 1, 0, 1, \ldots$, and $1, 0, 1, 0, \ldots$ are incomparable. There are also zero divisors. This doesn't cause any problems.

Why didn't we need the Transfer Principle? The Transfer Principle is a powerful schema of nonstandard analysis that says that any statement true about the real number system is true about the hyperreal number system and vice-versa. In elementary calculus and analysis, that is used chiefly to prove that the nonstandard definitions are equivalent to the standard definitions. For us, these equivalences are easy. Here's an example:

Definition 5.1 *f is* uniformly continuous *on C iff*
(Standard) $\forall \epsilon > 0\ \exists \delta > 0\ \forall x, y \in C\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
(Non–nonstandard) $\forall \mathbf{a}, \mathbf{b} \in C, \mathbf{a} \approx \mathbf{b}, \Rightarrow f(\mathbf{a}) \approx f(\mathbf{b})$.

Proposition 5.1 *The standard and the non–nonstandard definitions of uniform continuity are equivalent.*

Proof: Suppose the non–nonstandard definition holds and suppose we are given an ϵ for which there is no suitable δ. Then for each natural number n, choose a_n and b_n such that $|a_n - b_n| < \frac{1}{n}$, but $|f(a_n) - f(b_n)| \geq \epsilon$. Then we have $\mathbf{a} \approx \mathbf{b}$, but $f(\mathbf{a}) \not\approx f(\mathbf{b})$, a contradiction.
Suppose now that the standard definition holds and we are given $\mathbf{a} \approx \mathbf{b}$ and d, a positive real. By the standard definition, there is a δ such that $\forall x, y\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < d$. Then since $|a_n - b_n| < \delta$ (FAB), $|f(a_n) - f(b_n)| < d$ (FAB), and so $f(\mathbf{a}) \approx f(\mathbf{b})$. $\blacksquare_{\text{Prop. 5.1}}$

What happened to the Axiom of Completeness? We did use the completeness of the real line, but in a most innocuous and comprehensible form: we simply assumed that every infinite decimal corresponds to a real number.

And finally: **What happened to all the quantifiers?** In the case of uniform continuity, for example, we went from "$\forall \epsilon\ \exists \delta\ \forall x, y \ldots$" (logicians call this a Π_3 statement) to "$\forall \mathbf{a}, \mathbf{b}, \ldots$" (a Π_1 statement). We buried some of the quantifiers. We hid several in "$\mathbf{a} \approx \mathbf{b}$." If you recall, the definition of this is something like: "$\forall r > 0\ |a_n - b_n| < r$ (FAB) $\ldots$" which is really "$\forall r > 0\ \exists k\ \forall n > k \ldots$." Essentially, we simplified the definition by coding up the most difficult part.

6 Pedagogy

The non-nonconstructive infinitesimals presented here could improve the teaching of calculus. In both standard and "reform" calculus courses, rigor has been almost entirely omitted. Consequently, students are not asked to prove theorems until they have a fairly strong intuition for the subject and have met infinite sequences. This background makes non-nonstandard analysis very attractive.

The ideas in this paper may be especially appropriate for reform calculus sequences. At Smith College, for example, our two-semester introduction to calculus includes some vector calculus. This leaves room in the third semester for rigorous theory. Non-nonstandard infinitesimals make the usual theorems easier for students to understand. Many theorems can be left for students to prove themselves.

7 Some History

Infinitesimals were invented by the Greeks. They were also rejected by them. To Aristotle, there was no absolute infinity, only potential infinity. The distinction is very much like the difference between infinitesimals and limits. Archimedes used infinitesimals for intuition, then verified his results by proving them with (what we would call today) limits.

The calculus was first expressed in terms of infinitesimals. Its outstanding success overshadowed criticism, but there was criticism nonetheless. One can see in the work of Newton, hints of limits.

In the nineteenth century, Cauchy, Weierstrass, and others made infinitesimals unnecessary. The absolute infinites were replaced by limits. But linguistic habits didn't change. Consumers of calculus continued to talk in terms of infinitesimals. Infinitesimals did not disappear from calculus texts for over a hundred years.

Infinitesimals began to reappear in the twentieth century, but this time, well-defined. The idea of sequences as infinitesimals appears in a remarkable book, *The Limits of Science, Outline of Logic and Methodology of Science* by Leon Chwistek, painter, philosopher, and mathematician. The book, published in 1935, is not well known today (it is in Polish). Chwistek's definitions are similar to those presented here, though there are differences and limitations. His work foreshadows not only non-nonstandard analysis, but nonstandard analysis.

In [L1] and [L2], D. Laugwitz formulated the system of sequences described in this paper. Laugwitz used his "Ω–Zahlen" to investigate distributions and operators. An earlier paper by Schmieden and Laugwitz used a more primitive system with the idea of justifying the infinitesimals of Leibniz. Laugwitz's work was not carried further, possibly because the discovery of nonstandard analysis made real infinitesimals unglamorous.

In 1960, Abraham Robinson constructed nonstandard models of the real number system using mathematical logic ([R]). Robinson credits the papers of Laugwitz and Schmieden with some inspiration for his work. Nonstandard analysis requires a substantial investment (logic and the Axiom of Choice) but pays great dividends. Nonstandard analysis has been used to discover new theorems of analysis. It has been fruitfully applied to measure theory, brownian motion, and economic analysis, to name just a few areas. Attempts to reform calculus instruction along infinitesimal lines, however, did not have lasting success ([K]).

There are other systems of standard infinitesimals, Conway's surreal numbers, for example

[Co]. There are other systems for avoiding ϵs and δs (see Hijab's book, for a recent example, [Hi]). There may also be other rediscoveries of this system. The contribution of this paper is to place the structure in an algebra suitable for students of calculus.

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