Planar Minimally Rigid Graphs and Pseudotriangulations

November 21, 2003
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I. Rigid Graphs.

II. Robot Arms

III. Pseudotriangles

IV. Relationships
Part I: Rigidity of Graphs

Construct the graph $G$ in $\mathbb{R}^2$ using inflexible bars for the edges and rotatable joints for the vertices. This immersion is rigid if the only motions of this structure are trivial (rotations and translations).
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Total degrees of freedom: $2V$.
Adding an edge can remove a degree of freedom.
In the end you will still be able to translate and rotate.

Thus, if $G$ is rigid in $\mathbb{R}^2$ then $E \geq 2V - 3$. 
A graph is *rigid* in $\mathbb{R}^2$ if some embedding of it is.

Turns out, if some embedding is rigid then most are.
Aside: When the vertices are not in general position, can get non-rigid embedding. (rigid but not infinitesimally rigid)

We’ll be concerned with vertices in general position. (Generic Rigidity)
A graph is minimally rigid (2-isostatic) in $\mathbb{R}^2$ if removing any edge results in a graph that is not rigid.

These are not minimally rigid.
Theorem (Laman, 1970) A graph is minimally rigid if and only if \( E_G = 2V_G - 3 \) and for any subgraph \( H \) with at least 2 vertices, \( E_H \leq 2V_H - 3 \).

\[ V = 4 \text{ then } E = 2 \times 4 - 3 = 5 \]
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Current Applications of Rigidity.

- robotics (Streinu, Connelly, Demaine, Rote)
- geometrical properties of molecular conformations including protein folding (Whiteley, Thorpe)
- molecular modelling, (Bezdek, Streinu, and others)
Part II: Robot arms and Carpenter’s Rule

Given any planar embedding of a path can it be straightened?

Move edges without crossing them!
What about moving one embedding of a graph to another?

Trees?
Trees can Lock!

(Beidl, Demaine, Demaine, Lubiw, O’Rourke, Overmans, Robbins, Streinu)
Can any planar embedding of $C_n$ be convexified?
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Yes! Connelly, Demaine, Rote; nicer algorithm by Streinu.
Part III: Pseudotriangles

A pseudotriangle has 3 convex corners:
A pseudotriangulation is an embedding of a planar graph in which all interior faces are pseudotriangle. The exterior face is convex.
In *minimum* pseudotriangulation every vertex will be pointed.
A *pointed* vertex is incident to an angle $> 180^\circ$ (a *reflex* angle or *big* angle).
Shortest path (with given homotopy) turns only at pointed vertices. Addition of shortest path edges leaves intermediate vertices pointed.

→ geodesic triangulations of a simple polygon
[Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink 1994]
Lemma. If a face is not a pseudotriangle, then one can add an edge without creating a nonpointed vertex.

Go from a convex vertex along the boundary to the third convex vertex. Take the shortest path.

Can build a pseudotriangulation by adding shortest paths.
Can go from one pseudotriangulation to another by flipping edges.

[Kettering, Snoeyink, Speckmann]
Another way to **build a pseudotriangulation on a set of points** is to keep adding vertices of degree two. Order by x-coord.
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So a minimal pseudotriangulation has $2(n-2)+1=2n-3$ edges.
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Indeed,

Theorem: Minimal pseudotriangulations are minimally rigid. [Streinu]
Part IV: Relationships between these.

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Pseudotriangulations provide structure to convexify embedded cycles.
Algorithm for Convexifying:
Pseudotriangulate the polygon, including hull edges.
Remove one hull edge. Now have one-degree of freedom and its an *expansive motion*.

Open the object.
Repeat.

http://cs.smith.edu/streinu/Research/Motion/motion.html

http://cs.smith.edu/streinu/Research/Motion/Animation/mech1a.html
Part IV: Relationships between these.

So far:

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Next:

Are there rigid graphs that can not be embedded as pseudotriangulations?
Not every minimally rigid graph is planar:
Plane embeddings of minimally rigid graphs may not be pseudotriangulations
Pseudotriangulations are minimally rigid graphs. (Streinu)
Can every minimally rigid graph be embedded as a pseudotriangulation?
Pseudotriangulations are minimally rigid graphs. (Streinu)

Can every minimally rigid graph be embedded as a pseudotriangulation?

Yes!

Theorem: Every planar minimally rigid graph can be embedded as a pseudotriangulation. (HORSSSSSSW)
Characterizations of minimally rigid graphs.

**Theorem (Laman, 1970)** A graph is minimally rigid if and only if $E_G = 2V_G - 3$ and for any subgraph $H$ with at least 2 vertices, $E_H \leq 2V_H - 3$.

Algorithm? Not very efficient...
Characterizations of minimally rigid graphs.

**Theorem (Lovasz - Yemini, 1982)** A graph is minimally rigid if and only if adding any edge (including doubling an existing edge) results in a graph that is two edge-disjoint spanning tree.

Algorithm? Edmonds Matroid Partitioning Algorithm -polynomial in $V, E$. Run it $C(n,2)$ times!
Characterizations of minimally rigid graphs.

**Theorem (Crapo, 1992)** A graph is minimally rigid if and only if its edges are the disjoint union of 3 trees such that each vertex is incident with exactly 2 trees and in any subgraph the trees have different spans.

This characterization can be tested in one application of a modified Matroid partitioning algorithm.
Characterizations of minimally rigid graphs.

**Theorem (Henneberg, 1860):** A graph is minimally rigid if and only if it can be constructed using a Henneberg 2-sequence.
Begin with the graph consisting of 2 vertices and a single edge.

1. Add a new vertex of degree 2.
2. Delete an edge \( \{v_1, v_2\} \), and add a new vertex \( v \) adjacent to each of \( \{v_1, v_2\} \) adjacent to any additional vertex.
Use Henneberg to prove any minimally rigid graph can be embedded as a pseudotriangulation.

- Preprocess the vertices to find order they must be added using Henneberg moves.

Vertex order: $2, 3, 6, 7, 8, 1, 4, 5$

- Construct the PT. Maintain the pseudotriangulation at each step.
Adding a vertex of degree 2 to a pseudotriangle:
Henneberg type 2: Remove an edge: Get a pseudo-quad.
add a vertex of degree 3.

New vertex must also be pointed.
This always works.

This does not let us maintain the combinatorial structure of the embedding.

It is algorithmic, of order $O(n^3)$. 
Minor modifications allow us to:

- Predetermine the faces.

- Predetermine the outside face.
A different proof technique lets us:

Realize any COMBINATORIAL pseudo-triangulation
(pre-assign which vertex is pointed in each face.)

Combinatorial pseudotriangulations
There are Other Applications of Pseudotriangulations in Comp. Geom.

- Data structure for ray shooting [Chazelle, Edelsbrunner, Grigni, Guibas, Hershberger, Sharir, Snoeyink, ’94] and visibility [Pocciola and Vegter, ’96].
- Kinetic Collision detection [Agarwal, Basch, et. al. ’01]; [Kirkpatrick, Snoeyink, Speckmann ’00, ’02]
- Art Gallery Problems [Pocciola and Vegter, ’96] and [Speckmann and Toth, 01].
Current Work and Open Questions:

- Embedding Pseudotriangulations on a small grid. $O(n) \times O(n)$
- Graph theoretic properties of pseudotriangulations.
- Pseudotriangulations in 3-space.
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• Embedding Pseudotriangulations on a small grid. $O(n) \times O(n)$.?

• Graph theoretic properties of pseudotriangulations.

• Pseudotriangulations in 3-space. (Rigidity is not well understood!)
End.
Rigidity in 3-d.

Now every vertex has 3 degrees of freedom.
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**Theorem:** If a graph is minimally rigid in $\mathbb{R}^3$ then $E_G = 3V_G - 6$ and for any subgraph $H$ with at least 2 vertices, $E_H \leq 3V_H - 6$. 
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Now every vertex has 3 degrees of freedom.

There are 6 independent translations and rotations in $\mathbb{R}^3$.

**Theorem:** If a graph is minimally rigid in $\mathbb{R}^3$ then $E_G = 3V_G - 6$ and for any subgraph $H$ with at least 2 vertices, $E_H \leq 3V_H - 6$.

Converse is false...