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#### A MODIFIED MOORE METHOD FOR TEACHING UNDERGRADUATE MATHEMATICS

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**1. Introduction.** The late R. L. Moore made famous a method of teaching mathematics which has had a profound influence on educators [3]. Professor Moore's most important departure from age-old teaching practices was that he gave no lectures. Instead he gave his students theorems to prove and required them to work on the proofs until they were correct. Professor Moore selected

those who would take his graduate course from a list of students eager to participate, and many of his students became excellent productive mathematicians.

When the "Moore method" was attempted by others at the undergraduate level, however, the results often were disappointing. Instructors of undergraduates usually cannot handpick their students. Moreover, undergraduates normally have not had the experience proving theorems and writing mathematics necessary for meaningful progress without help from an instructor. Finally, the Moore method sometimes creates an unhealthy atmosphere of competition and isolation among students.

Recently, Professor Moore's philosophy of shifting the focus from instructor to student has been incorporated in a wide variety of teaching methods, which are more successful for undergraduate courses. (See [4] and [5].) The method I describe here has worked very well and is the most comprehensive program I have seen for introducing a Moore-type method to undergraduate teaching.

Although this Modified Moore method was developed to teach mathematics, it is based on principles that apply to the teaching of most subjects.

1. Students understand better and remember longer what they discover themselves than what is told to them.
2. People master an idea most thoroughly when they teach it to someone else.
3. Effective writing and clear thinking are inextricably linked.

The following objections usually arise when one thinks about teaching methods devised to put these principles into practice.

1. The truths taught in most courses took great thinkers many years to discover. We cannot expect our students to discover them all in one semester. Further, the amount of the material we cover in our courses is at a minimum and should not be reduced.
2. It might be wonderful training for students to try to teach what they have just learned, but pity the poor student who must learn from such a teacher.
3. Yes, we wince at some of the writing we get from students, but we are not teaching writing courses. We do not have the time or the training to teach writing. The serious students will learn to write when they get to graduate school and read journals. If their writing is fuzzy, we can test their understanding with short-answer questions.

The Modified Moore method, based on the principles stated above, answers these objections. In brief, the method divides a class into small groups, each of which is responsible for a weekly question. Over the course of a week each small group will study and answer the question, write a short paper presenting the answer, and prepare to teach the question and its answer to the rest of the class. The questions and their answers contain all the ideas covered in the course.

**2. Preparation and Scheduling for the Course.** The first task for the instructor is to break the course material into a list of *questions*. These may resemble questions asked on a take-home examination at the end of the course. Some examples of questions for a course in real analysis are provided in the Appendix. An average student with some coaching from the instructor should be able to understand a question and write an answer to it in one week. There should be enough questions so that any student who can answer all of them thoroughly will have mastered all the material for the course. The number of questions should be about two or  $2\frac{1}{2}$  times the number of weeks allotted for the course. This number will provide weekly allotments of two or three questions per week and allow three weeks to be reserved for organizing the class and giving examinations.

The next task is to compile a list of *basics* that the students must understand to answer the questions. These are usually definitions or axioms or theorems from other courses. In the

Appendix I have provided examples of basics as well as questions.

With the Modified Moore method the class is scheduled to meet as an entire class once a week for *class meetings* of 90 to 110 minutes. It does not work well to substitute two 50-minute class meetings for the one longer meeting, since ample time is required for students to ask questions about small points. Furthermore, the questions for the week are often intimately related and are better discussed in a single meeting. Two other hours on two separate days each week are then scheduled for *coaching sessions*.

**3. Organizing the Class and Assigning Work.** The first day of class is devoted to a thorough explanation of the method, its objectives, and what is expected of the students.

The class is then divided into teams of two or three students each. Teams of three work best. These do not change until midterm when all teams must change for the second half of the term. New teams are assigned to avoid those consisting only of weak or only of strong students.

At the end of each class meeting the instructor assigns questions for the next class meeting. Several teams may be assigned the same question. After assigning the questions, the instructor announces the schedule for the first coaching session.

The assignment is to work as a team to: 1) understand the question, 2) find the answer, and 3) write a short paper presenting the question and its answer. This paper will be duplicated and distributed to all members of the class. In addition, each team must be prepared to explain to the class its question and the answer it has found. All teams are expected to be prepared for each class meeting, although only one team per question will be called upon to make an oral presentation.

**4. The Coaching Sessions.** The first coaching session follows the class meeting by one or two days. Each team is required to come during that part of the first coaching session dealing with its question. If several teams are dealing with the same question, all of those teams come at the same time. Each team is expected to have met once prior to this session to try to understand the basics and the question and to formulate ideas about the answer.

When students have discovered the answer before the session, the instructor simply listens to the team's answer and provides whatever help is necessary to make the team's explanation very clear. When a team comes in a state of confusion, on the other hand, the instructor must do some explaining. In the 20 or 30 minutes devoted to each question, he must try to lead the students to the heart of the matter, using a Socratic style as long as time permits it. Sometimes this means asking one or two questions to help students understand how they almost had the idea themselves. At other times it means giving a complete explanation, and hoping that the students will make a discovery on some other questions. I try to make sure that at least one member of each team has a basic understanding before the teams leave.

The second coaching session follows the first by three days or more. During those days the students prepare drafts of their papers. The teams meet and discuss the work and a team might even agree on a rough written draft of its paper. Each member must then write his or her own draft without consultation.

All teams gather for the second coaching session during which students exchange their drafts with their teammates for peer review. The reviewers pay close attention to content, clarity and style, and make suggestions for improvement.

While this peer review is taking place, the instructor selects drafts at random and reviews them in front of the team. This provides at least one team member with professional comment on a draft, and at the same time gives each member a chance to watch how the instructor reviews. I reserve some time to speak to the class as a whole about common errors or misconceptions found during my reviews.

**5. The Weekly Papers.** Each paper must consist of a clear, concise, cogent answer to the team's question for the week, written in a style appropriate to the subject for the course, and with careful attention to syntax, punctuation and correct usage of notation commonly used by

professionals in the subject.

Each paper must begin with an abstract of two or three sentences revealing the key idea in the paper. Students perform a task important for clear understanding when they distill a long argument into two or three essential sentences.

Each team decides how the paper is to be written. Sometimes the students divide a paper into sections with each member responsible for writing a section. Sometimes a single student writes the entire paper; however, that student may not write another one until the other members of the team have also written an entire paper.

**6. The Class Meetings.** At the beginning of the class meeting each team passes a copy of its paper out to all students. Two copies go to the instructor, one to be corrected and returned. The team called upon for the first question for the week then makes its presentation. Each team must plan its presentation and allow time for questions from the class. Usually each member is responsible for one part of the presentation.

The instructor's role during the presentation is a delicate matter. If there is a point that seems unclear to me, I wait for a student to ask about it; but if no one does, then I ask. I am willing to help a team clarify an explanation, but only after I sense that no one else can do it. With this method students are much less willing to let the instructor dominate discussion than in more conventional classes. They have invested much time and energy in their preparation, and are anxious to explain their ideas clearly to their classmates.

**7. Evaluating Student's Work.** The instructor should try to read each team's paper carefully and return the papers the day after they are received. He should correct grammar and style as well as content.

The Modified Moore method calls for two comprehensive written examinations; one at midterm and one at the end of the course. Students take these as individuals, not as teams. Everyone is responsible for all the material discussed in class meetings, including material the instructor may have provided on the spur of the moment to clarify or amplify an idea.

**8. The Method at Smith College.** I have successfully used the Modified Moore method to teach courses at all levels at Smith: a course on "The Infinite" for nonmathematics majors, a course on Hilbert spaces for senior mathematics honors students, as well as standard intermediate-level courses. The method has worked well with classes having as few as 5 students and as many as 25. I suspect that it would not work very well for classes much over 25 because the coaching sessions would be too large to allow time for the Socratic approach.

Students are encouraged to consult appropriate text books, which I place on reserve at the library. Sometimes I even cite the pages of a book that contain the answer to a question for the week. To read the answer a student must understand the author's notation and refer back to results the author has proved earlier to see if those results have been proved in our class.

I grade papers and oral presentations either "0", "1", or "2". Most grades are "1", which indicates satisfactory work. A grade of "2" indicates an unusually impressive piece of work. A grade of "0" indicates that the paper missed the idea altogether. Every member of a team receives the same grade for that team's paper, but members receive individual grades for their part in an oral presentation.

**9. Conclusions.** The objections cited in section 1 can now be refuted.

The Modified Moore method incorporates the principle of student discovery without undue sacrifice in amount of material covered. While each student grapples each week with only one question, other students are dealing with other questions, so that the total amount of material covered by the class in one week is roughly equal to that which would be covered by lectures. I am convinced by students' performance on examinations that the total amount of material students master in courses taught by the Modified Moore method is at least as great as the amount they master in lecture courses.

The method effectively raises the level of communication between students. Since all students are responsible for all questions on examinations, the presenting teams try hard to teach and the other students demand the clarity they need in order to learn.

Since students write every week and receive prompt evaluation of their writing, they make substantial improvement. They also learn to appreciate the connection between understanding a subject and writing about it.

Furthermore, most students respond well to the responsibility placed on them by the Modified Moore method. They are willing to work hard for a teacher who pays attention to their thinking and writing. I find that the percentage of students who drop courses taught by the Modified Moore method because they are burdensome is no higher than the percentage who drop other courses for that reason. Most students are stimulated by the change from passive to active learning.

**Appendix.** Below is a sample syllabus and the basics and questions for class meeting number six for the course Introductory Real Analysis taught for four years at Smith College by the Modified Moore method. The course is for sophomore and junior mathematics majors. The prerequisites are Multidimensional Calculus and Linear Algebra. There are ten weeks available for class meetings out of the 13-week semester and the total number of questions for the course is 27.

#### Week

1. Orientation
2. Class meeting 1: Metrics; completeness of the reals; open and closed subsets of the reals.
3. Class meeting 2: Limit points; Bolzano-Weierstrass Theorem.
4. \_\_\_\_\_ 3: Continuity of functions, introduction.
5. \_\_\_\_\_ 4: Continuity of functions, sums, products, compositions, examples.
6. \_\_\_\_\_ 5: Differentiability of functions.
7. \_\_\_\_\_ 6: Cauchy sequences of reals; pointwise and uniform convergence of sequences of functions.
8. Examination
9. Class meeting 7: Theorems on uniformly convergent sequences of functions; Weierstrass  $M$ -Test.
10. \_\_\_\_\_ 8: Analytic functions; Taylor's Theorem.
11. \_\_\_\_\_ 9: Lebesgue measure, introduction.
12. \_\_\_\_\_ 10: Lebesgue integral, introduction.
13. Examination

#### Basics for Class Meeting Six. (Week 7)

DEFINITIONS: a Cauchy sequence of real numbers, the limit of a sequence of real numbers, the pointwise limit of a sequence of functions, the uniform limit of a sequence of functions.

#### Questions for Class Meeting Six. (Week 7)

1. Show that the sequence of functions  $f_n(x) = x^n$  converges pointwise but not uniformly on  $[0, 1]$  to the function

$$h(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

2. Show that the sequence of functions  $f_n(x) = 1 - x^n$  converges uniformly on  $[-p, p]$  for any  $p$  with  $|p| < 1$ .

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