How much string do you need to tie your shoelaces?

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Smith College

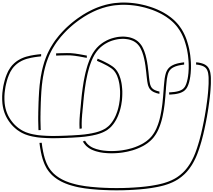
Science in the Center 10 minute talk

Outline

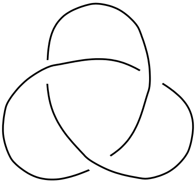
What is a knot?

Definition

A **knot** K is an embedding of a circle in $\mathbb{R}^3$. (Intuition: a smooth or polygonal closed curve without self-intersections.)
A **link** is an embedding of a disjoint union of circles in $\mathbb{R}^3$.



Hopf Link



Trefoil knot

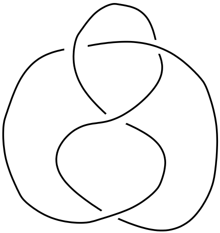
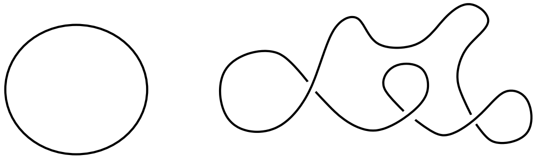


Figure 8 knot

When are two knots the same?

- K_1 and K_2 are **equivalent** if K_1 can be continuously moved to K_2. (Technically, they are ambient isotopic – knots are embedded in space.)
- A knot is **trivial** or **unknotted** if it is equivalent to a circle.



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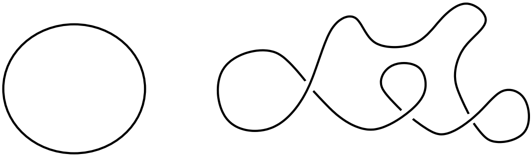

How can you tell if two knots are equivalent?

Idea

Use topological invariants denoted by I. If K_1 is equivalent to K_2 then $I(K_1) = I(K_2)$,

- Crossing number (knot tables)

$$cr(K) = \min_{K \in [K]} \left(\min_{\text{directions}} (\#\ \text{crossings of } K) \right)$$

- Alexander or Jones polynomial
- Knot group $\pi_1(\mathbb{R}^3 \setminus K)$

Principle

If $I(K_1) \neq I(K_2)$ then K_1 is not equivalent to K_2. Knots are distinguished using invariants.

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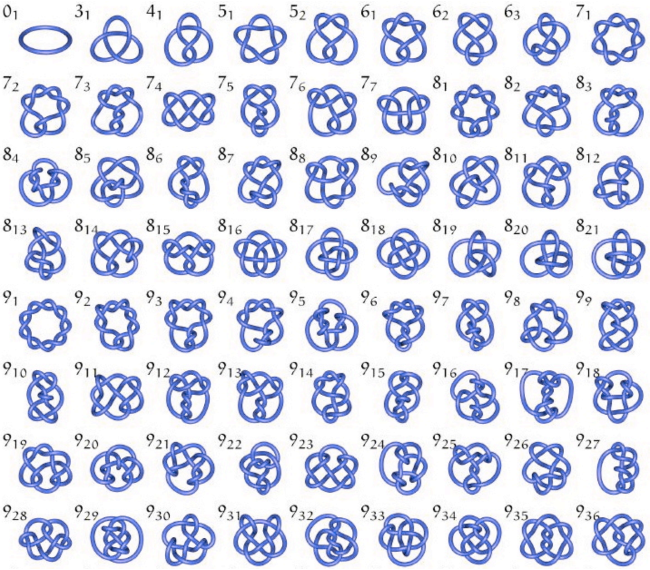
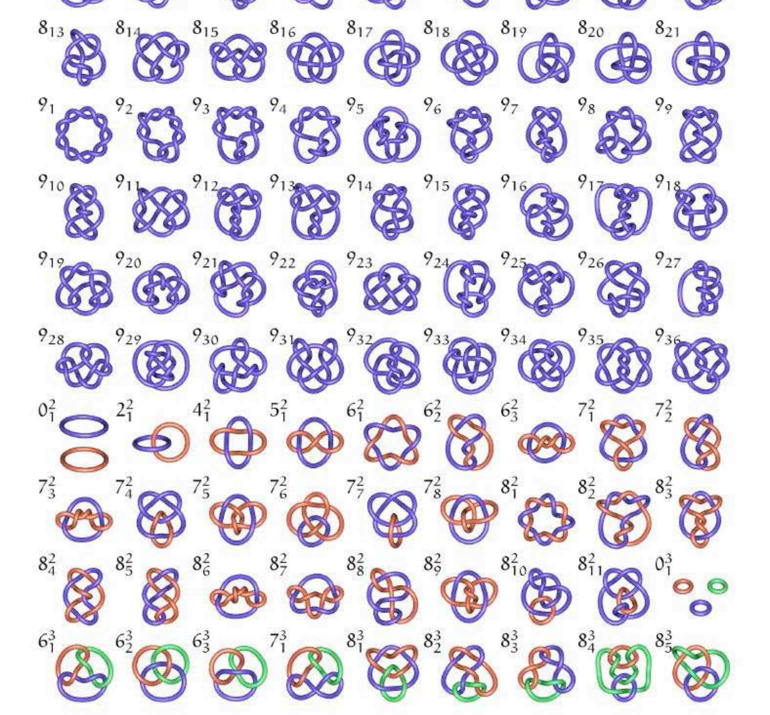
Knot and Link Table




8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}
0^2_1 2^2_1 4^2_1 5^2_1 6^2_1 6^2_2 6^2_3 7^2_1 7^2_2
7^2_3 7^2_4 7^2_5 7^2_6 7^2_7 7^2_8 8^2_1 8^2_2 8^2_3
8^2_4 8^2_5 8^2_6 8^2_7 8^2_8 8^2_9 8^2_{10} 8^2_{11} 0^3_1
6^3_1 6^3_2 6^3_3 7^3_1 8^3_1 8^3_2 8^3_3 8^3_4 8^3_5

The ropelength problem

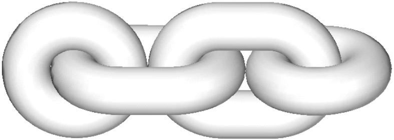
Goal

Minimize ropelength — the length of a knot which has an embedded tube of fixed radius around it.

Tight knots are those knots which minimize ropelength.

Question

What do tight knots look like? What is the ropelength of a tight knot?



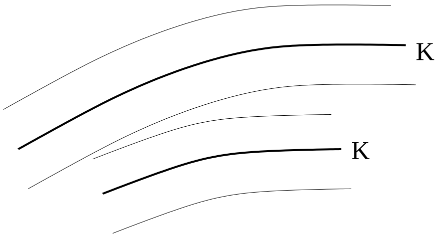
Definition

The **ropelength** $R(K)$ of a knot K is the quotient of its length over thickness $R(K) = len(K)/\tau(K)$.

Definition

The **thickness** $\tau(K)$ is the **radius** of the largest embedded normal tube around the knot K.

Conditions: local (curvature), global (self-intersections).



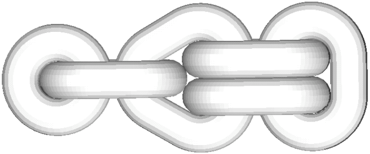
- In 2002 Cantarella, Kusner and Sullivan proved that there is a ropelength minimizer in any link type. They also described a family of tight links.

Theorem (Combined results of many people)

If $\mathcal{L}$ is a link type with minimum crossing number n and minimum ropelength $R(\mathcal{L})$, then

$$\left(\frac{4\pi}{11}n\right)^{3/4} \leq R(\mathcal{L}) \leq C_1 n^{3/2}$$

- It is conjectured that $R(\mathcal{L}) \leq C_2 n$.



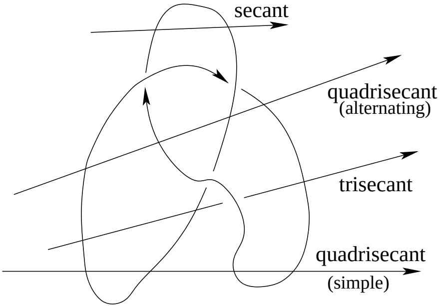
- There are many links for which the minimum ropelength is not known.
- The minimum ropelength of any *knot type* is not known.
- Numerical simulations have shown that the trefoil is the shortest possible knot with length about 16.372.

Theorem (D-,Diao, Sullivan 2006)

Any nontrivial knot has ropelength at least 15.66.

(So for the trefoil, the theory is pretty close to the numerics.)

Quadrisecants



Secant = 2-secant

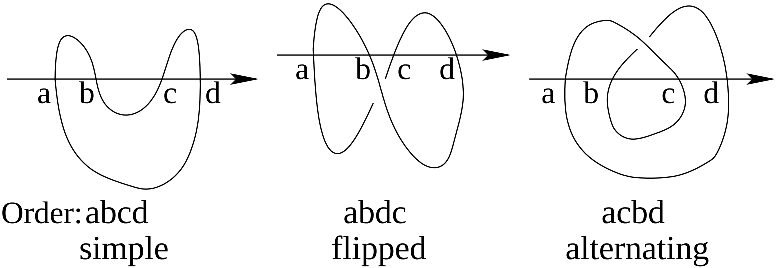
Trisecant = 3-secant

Quadrisecant = 4-secant

Definition

An n**-secant line** is an oriented line in $\mathbb{R}^3$ which intersects K has at least n places. An n**-secant** is an ordered n-tuple of points in K which lie on an n-secant line.

- Three types of quadrisecants: simple, flipped and alternating.
- Determined by comparing the order of *abcd* along the line and along the unoriented knot.



Theorem (D–,2004)

Every nontrivial tame knot in $\mathbb{R}^3$ has at least one essential alternating quadrisecant.

Strategy

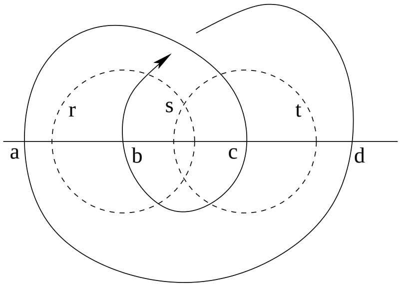
- *Assume knots have unit thickness, then ropelength is just the length of the knot or link (the core curve).*
- *As the knot is thick, the knot has some extra geometry.*
- *Use this and quadrisecants to get a lower bound for ropelength.*

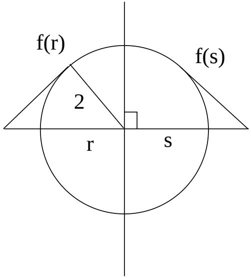
Computations

- Given quadrisecant *abcd*, define

$$r := |b - a|, \qquad s := |c - b|, \qquad t := |d - c|.$$

- Find $Len(K)$ in terms of r, s and t.
- Let $f(r) := \sqrt{r^2 - 4} + 2\arcsin(\frac{2}{r})$.
- Length $\gamma_{ac} \geq f(r) + f(s)$, length $\gamma_{da} \geq f(r) + s + f(t)$.





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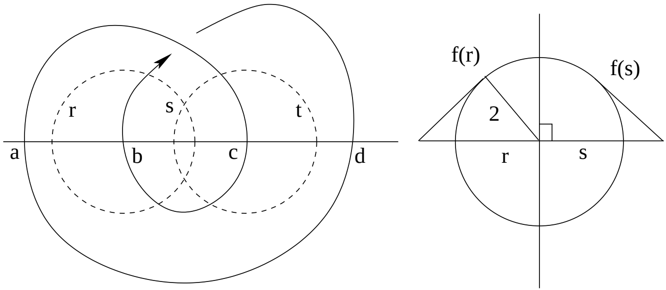

The results

Theorem (D-,Diao, Sullivan 2006)

Any nontrivial knot has ropelength at least 15.66.

Recall the tight trefoil knot was estimated to have ropelength 16.372, we are pretty close in this case!

Note the ropelength of knots with

- simple essential quadrisecants is at least 15.94,
- flipped essential quadrisecants is at least 13.936.

Thank you!

Further reading:

- Colin Adams *The knot book: an elementary introduction to mathematical theory of knots*
- A.B. Sossinsky *Knots: mathematics with a twist*
- Charles Livingston *Knot theory*
- Kunio Murasugi *Knot theory and applications*

Definition

Let α, β, γ be disjoint arcs from *p* to *q*. Let $X := \mathbb{R}^3 \setminus (\alpha \cup \gamma)$ and let $h(\alpha)$ be a parallel curve to $\alpha \cup \beta$, chosen so that $\alpha \cup \gamma$ has linking number zero with δ. Then (α, β) is **inessential** if δ is also null homotopic in *X*. We say (α, β) is **essential** if it is not inessential.