Research statement

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My research focuses on geometry, topology, and dynamical systems, with a particular emphasis on translation surfaces and complex dynamics. I explore invariants that help to classify these objects and processes and to determine their properties. These invariants may be geometric, combinatorial, topological, or algebraic. In many cases, they reveal hidden structures that illuminate the nature of these objects and processes. Often, they provide an accessible gateway into current research trends that can function pedagogically as well as technically.

Some of the areas of mathematics that relate to my research are complex analysis, hyperbolic geometry, algebraic geometry, group theory, number theory, graph theory, Lie algebras, ergodic theory, and homogeneous spaces. These areas serve to provide both motivation and tools for the study of translation surfaces and dynamics with complex variables. I will sketch some of my past and ongoing work, along with the ways I have used it to connect with students and engage them in active mathematical research.

1 Translation surfaces

A translation surface is a surface equipped with an atlas of charts whose transition maps are translations in $\mathbb{R}^2$. In other words, a translation surface can be assembled from polygons in $\mathbb{R}^2$ by identifying pairs of edges in such a way that each identified pair is parallel and has the same length. An example is a square whose top and bottom edges are identified, and whose right and left edges are identified; this results in a surface that is topologically a torus. Translation surfaces appear naturally in the study of several dynamical systems, such as polygonal billiards, interval exchange transformations, homeomorphisms of surfaces, and geodesics in moduli spaces.

As part of my thesis work [B1, B2, B3], I considered triangulations of translation surfaces. A translation surface may be expressed in many different ways as a union of polygons with edge identifications. Because triangles are the simplest kind of polygon, it is worth considering how a given translation surface may be decomposed into triangles. A Delaunay triangulation is one in which the sum of the angles opposite each edge is less than or equal to $\pi$. (This definition is inspired by the usual notion of Delaunay triangulation for a set of points in $\mathbb{R}^2$.) As the translation surface is deformed (for instance, by applying a linear transformation to the coordinate charts of the surface), the Delaunay triangulation changes, and surfaces may be grouped according to the topological types of their Delaunay triangulations.

I have presented these ideas several times at undergraduate seminars. The concrete nature of the problem of triangulations, as well as the intriguing blend of geometry, topology, linear algebra, and combinatorics, makes it a good illustration of how mathematical research is often motivated by relatively simple questions that have not yet been answered, and it generally occurs at an intersection of established areas. There are many questions about triangulations of surfaces that would provide accessible research projects for undergraduates, particularly ones with an interest in programming, such as studying certain analytic properties that triangulations carry.

A homothety surface is similar to a translation surface, except that transition maps are allowed to rescale as well as translate (but rotations are not permitted). An example is an annulus whose inner and outer boundary circles are identified by scaling around the center; this again results in a topological torus, but its geometric structure is very different from that of the torus obtained from a square. Homothety surfaces...
have to date received less attention than translation surfaces, but they have applications in many of the same areas.

While at Smith, I have started research with students on homothety surfaces, to see how their behavior differs from that of translation surfaces. In my first research group, which I directed last spring, we considered a connected sum of two tori, one formed from a square and the other from an annulus. Our research focused on straight-line trajectories on this new surface; to aid our explorations, the students wrote two programs that computed the positions of trajectories over time, one in Matlab and one in Mathematica. We found that the “interesting” trajectories are those that cross between the two component surfaces infinitely often, and we collected data on the properties of periodic trajectories, which eventually close up into circles [BOP]. In the future, I would like to work with students to answer further questions about these surfaces, such as interpreting the data already collected, and investigating how the possible behaviors of trajectories change as the surface is deformed.

Translation surfaces of finite genus, with finitely many singularities, have been widely studied since the 1980s. However, surfaces of infinite type appear naturally in several situations—such as wind tree models of particle scattering, billiards in irrational polygons, and horseshoe-type homeomorphisms of the sphere—and these have been gaining interest in recent years.

This past year, I was on the organizing committee for the International Conference and Workshop on Surfaces of Infinite Type, held in Morelia, Mexico, from July 29–August 2, 2013, as a satellite activity of the first Mathematical Congress of the Americas. We gathered 50 participants from all over the world, including many young participants with a budding interest in the subject, as well as senior researchers who expressed their appreciation for a conference on this topic.

I have also published multiple papers in the area of infinite-type translation surfaces. During my thesis work, I discovered a particular example of an infinite-type surface that extends a well-known family of compact translation surfaces [B5]. I clarified the notion of “infinite type” by differentiating among four different finiteness conditions, and determining their interdependence. I also showed that any of these finiteness conditions, not necessarily just the classical case of finite genus with finite area, imposes restrictions on the set of possible symmetries of the surface [B4]. In joint work with F. Valdez of the Universidad Nacional Autónoma de México, I constructed an invariant to study neighborhoods of certain kinds of singular points for infinite-type surfaces, which we dubbed “wild singularities” [BV].

Infinite-type surfaces remain an active area of interest for me. They connect naturally with the homothety surfaces described earlier, because any homothety surface has a covering which is a translation surface, possibly of infinite type. As mentioned, they also arise on their own in certain dynamical systems and as limits of classical translation surfaces.

2 Complex dynamics

The field of complex dynamics, as the study of holomorphic maps $\mathbb{C} \to \mathbb{C}$, was initiated by Fatou and Julia in the early 20th century, and was reinvigorated in the 1970–80s by work of Douady–Hubbard, Sullivan, Thurston, and others. It now encompasses a wide range of dynamical systems, including holomorphic and rational self-maps of complex manifolds and complex varieties. It has also become a prominent source for mathematical imagery in the popular consciousness over the past thirty years, particularly in the form of Julia sets (defined below) and the Mandelbrot set.

I have been studying polynomial self-maps, called endomorphisms, of complex projective spaces $\mathbb{C}P^k$ (for example $\mathbb{C}P^1$ is the Riemann sphere $\mathbb{C} \cup \{\infty\}$). Given an endomorphism $f : \mathbb{C}P^k \to \mathbb{C}P^k$, its Julia set is the set of points for which $f$ behaves “chaotically” on every neighborhood, meaning that the long-term behavior of points under iterates of $f$ is sensitive to the initial starting point. Julia sets hold particular dynamical interest in the study of endomorphisms.

Some examples of simple dynamical behavior that still exhibit chaos are provided by the Chebyshev
polynomials $T_n$ for $n \geq 2$, which act as maps $\mathbb{CP}^1 \to \mathbb{CP}^1$. These are defined by the functional equations $T_n(z + z^{-1}) = z^n + z^{-n}$; for example, $T_2(w) = w^2 - 2$. The Julia set of any Chebyshev polynomial is the segment $[-2,2]$ in $\mathbb{C}$. These functions $T_n$ have been studied in a variety of contexts, both dynamical and non-dynamical. They were generalized to all dimensions in the 1980s by Veselov and Hoffman–Withers by using root systems of Lie algebras ([B7] contains a precise definition). Little has been uncovered about the dynamical and analytic properties of these higher-dimensional analogues, however.

I have found that in the higher-dimensional cases of Chebyshev-like maps, the geometry of the Julia sets is related to classical curves like hypocycloids and rational normal curves [B6, B8]. This is a new phenomenon in the field of complex dynamics (where known Julia sets are generally either fractals or unions of linear subspaces), and also gives fresh motivation to the study of these curves. I recently presented some of these results at Smith’s weekly math talk, and the students responded with enthusiasm. Later I learned that several of them became interested in studying complex analysis. Again, I believe that the blend of old and new mathematics, at an intersection of multiple fields, presents students with an enticing foray into the ongoing process of discovering mathematics.

I have also studied a particular one-parameter family of quadratic endomorphisms in dimension 2. These maps arise from a certain “rock-paper-scissors” dynamic that had been previously studied by biologists and game theorists; adding the complex dimension significantly enriches them. In joint work with M. Lyubich of Stony Brook University, I examined the critical points and critical values of these maps and found a close connection between the critical points that escape to infinity and topological properties of the set of all such escaping points [BL]. When these maps are restricted to the real plane $\mathbb{R}^2$, they present a puzzling set of properties. Because on the real plane the maps are easily visualized and explored concretely, this could also be an area of potential research for advanced students with interest in topology and dynamics.

I am committed to pursuing research in collaborative environments, such as at conferences and in student research groups. I believe that teaching and research complement each other. On the one hand, the act of creating new mathematics informs pedagogical practice by keeping the subject alive and placing the educator in the mindset of discovery, to guide students into similar experiences. On the other, teaching forces the re-evaluation of even advanced topics from an elementary perspective, providing clarity and new insights for the researcher.

References


