Where Geodesics Go to Die

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Abstract

Given the dimensions \((a, b, c)\) of a box, we investigate geodesics that start at one corner and move on the surface at angle of 45° from the edges.

Our results and conjectures concern whether or not the geodesic ends at a corner, and if it does, its destination.

1 A Leisurely Introduction

The story begins with the old and comparatively simple situation of a ray bouncing in a rectangle, striking the sides at 45°. Given a rectangle with dimensions \((a, b)\), a ray starts at the lower-left corner and bounces off the sides,

\(^1\)The authors would like to thank Michael Henle for discussions and fruitful ideas.
until it eventually reaches a corner. Which corner it reaches and the length of its path depends on the dimensions $a$, $b$, of the rectangle. It’s not hard to show that the ending corner is

- A, if $a$ is more even than $b$,
- B, if $b$ is more even than $a$,
- C, if $a$ and $b$ are equally even,

where the *evenness* of a number is measured by the number of 2’s in its prime factorization.

If the ratio of the dimensions of the rectangle is not rational, the path never ends.

The generalization of this problem to a ray bouncing inside a box, a rectangular parallelepiped, is what you would expect and not particularly interesting. Suppose, however, that instead of traveling through the interior of the box, we travel on the surface of the box. The path is a geodesic, by which we mean that it is the shortest path (restricted to the surface of the box) between any two points on the path that are sufficiently close. It’s like a ribbon winding around the box.

As with the billiards problem on the rectangle, we can ask if the path reaches a corner. If it does, we can ask which corner it reaches. The situation is far more
complicated here, and there are some surprises. In particular,

1. The pattern of corners reached generates a fractal resembling some well-known gaskets.

2. By contrast with the rectangular case, it is possible for a geodesic to reach a corner even when the dimensions are not rational or rationally related. For example, the box with dimensions $\sqrt{2}$, $\pi$, $e$.

3. There are reasonably nice answers regarding the corners and as in the case of the rectangle, evenness is involved.

The connection to rectangles is underlined by the case where the height of the box is zero. Then the geodesic looks just like a ray bouncing in a rectangle—except that the box still has a top and a bottom.

The question we investigate here is connected to many fields of research: billiards on polygons, geodesics on polyhedra, translation surfaces, cutting sequences, symbolic dynamics, Teichmüller theory, generalized continued fractions, to name a few. In general, current research in these areas deal with infinite paths, infinite sequences, global issues. Geodesics are studied, but for geodesics on a box, for example, the focus is on paths that avoid corners, that continue infinitely in both directions. Our interest here is on local issues, on geodesics that start and end at corners.

2 The Color Triangle

Imagine a box in the first octant of $\mathbb{R}^3$, with the starting corner at the origin. Let $(a, b, c)$ be the dimensions of the box.

It will be convenient later to identify the eight corners of the box as 000, 100, 010, 011, 110, 101, and 111. The colors will also be useful.
Our paths will always start at the origin on the $xy$-plane. If the path for dimensions $(a, b, c)$ end at a corner, let $E(a, b, c)$ be that corner, the “destination”. Let $L(a, b, c)$, measured in units of $\sqrt{2}$, be the length of the path, so for example we have easily the following:

**Proposition 1** For any $r, s > 0$,

\[
E(r, r, s) = 110 \\
L(r, r, s) = r
\]

\[
E(r, r + s, s) = 111 \\
L(r, r + s, s) = r + s
\]

\[
E(r + s, r, s) = 111 \\
L(r + s, r, s) = r + s.
\]

To get an idea of what is happening in general we can interpret the dimensions of a rectangle, when normalized, as the barycentric coordinates of a point in an equilateral triangle. We have, for example, that $E(3, 3, 5) = 110$. We can use

\[
\left(\frac{3}{3+3+5}, \frac{3}{3+3+5}, \frac{5}{3+3+5}\right) = \left(\frac{3}{11}, \frac{3}{11}, \frac{5}{11}\right)
\]

to locate a point in an equilateral triangle with altitude 1. We can color the point cyan, the color of vertex $110 = E(3, 3, 5)$.
If we do this for a great many integral triples, we get what we call the "color triangle." It resembles a colorful Sierpinski gasket.
The pattern of triangles (without the colors) is connected to the Rauzy gasket, a figure homeomorphic to the Sierpinski gasket. The Rauzy gasket is defined in terms of ternary “episturmian sequences.” Episturmian sequences are defined in terms of the complexity of their finite subsequences. These in turn are related to cutting sequences, sequences that chart, for a given geodesic, the order in which edges are crossed.

The center of the color triangle is key; what happens there repeats infinitely in smaller, distorted versions. The points in the center correspond to what we will call triangular numbers.

**Definition 1** A trio of positive numbers, $a, b, c$, is triangular if none of them is greater than the sum of the other two.

The plan of the rest of this paper is as follows. In Section 3, we analyze boxes whose dimensions are triangular. In Section 4, we extend the analysis to boxes of arbitrary dimensions. In Section 5 we show some boxes where the path is infinite. In the last section we discuss some of the questions left open.

### 3 Triangular Dimensions

We start by following the path in the first example where the dimensions were $(6, 10, 5)$. The path returns to the original corner.

We can see how this works by laying the path out in the plane. We call the following diagram the “unfolding” of the box.

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1We are indebted to Edmund Harriss for pointing this out to us.
What makes this path miss certain corners and return to the starting point are the inequalities \( a + c > b \) and that \( b > a, c \). This gives us:

**Proposition 2** For any triangular \( a, b, c \) with either \( a > b, c \) or \( b > a, c \),

\[
E(a, b, c) = 000 \\
L(a, b, c) = a + b + c.
\]

Proposition 2 holds not just for natural numbers but for all positive real numbers. In particular, it gives us that \( E(\sqrt{2}, \pi, e), E(\pi, \sqrt{2}, e), E(\pi, e, \sqrt{2}) \), and \( E(e, \pi, \sqrt{2}) \) are all 000. Another consequence is that at least two-thirds of the color triangle is black, the color of corner 000.

The situation where \( c \geq a, b \) is more complicated, but we can prove:

**Proposition 3** For any positive integers \( a, b, c \), \( L(a, b, c) \) is finite.

**Proof:** If the dimensions of the box are whole numbers, the path will always hit an edge at an integral distance from the corners at the end of the edge. That means that there are only a finite number of points on an edge where the path can pass. The path can hit an edge point at most twice (crossing itself at 90°). It should be clear that the path cannot join itself. Thus it must eventually reach a corner.

\[\blacksquare\]
The case for $c \geq a, b$ resembles the solution for bouncing in a rectangle in that it depends on evenness.

**Proposition 4** For any triangular $a, b, c$ with $c \geq a, b$,

- $E(a, b, c) = 101$ if $c - a$ is more even than $c - b$
- $E(a, b, c) = 011$ if $c - b$ is more even than $c - a$
- $E(a, b, c) = 110$ if $c - a$ and $c - b$ are equally even.

**Proof:** Note first that in view of Proposition 2, any path starting on one of the four walls of the box (instead of the bottom or top) must end at the corner where it began, on the other adjacent wall of the box. Every path that ends on a wall is part of a loop that begins on a wall, so no path that ends on a wall can begin on the bottom or top. A path starting on the bottom face must therefore end running along the bottom face or end on the top face.

Thus, a path starting on the bottom face must end running along the bottom face or end on the top face.

To see what is going on when we start along the bottom, we view the box from above.

Our plan is to impose a checkerboard pattern on the bottom and top faces of the box to see which corners are possible destinations and which are not. We use the same pattern for top and bottom with the lower-left corner pink. As an example, consider a box with dimensions $(7, 5, 10)$. 


Seen from above, the path starts like this on the floor,

then continues
like this as it
climbs up the
walls of the box.

The path continues on the top of the box,

and then
down the walls
of the box.

Notice that on the bottom the path moves SW-NE or NE-SW and on the top the path moves SE-NW or NW-SE.

Claim 1 Seen from the top, as the path moves on the walls it always passes exactly two corners.

Proof of Claim 1: It must pass at least one corner, since $c > a, b$. It can’t pass three because $c < a + b$. But if it ever passed just one corner.

then by symmetry it would have earlier passed by just one corner (note that on the left $p + c + q = a + b$, and on the right, $p + c + r = a + b$, so $q = r$).
Thus there can’t be a first time to pass just one corner.  

**Claim 2**  *Seen from the top, the path always moves SW-NE or NE-SW on the bottom and SE-NW or NW-SE on the top.*

*Proof of Claim 2:* At the start the path is on the bottom and moving SW-NE. From the previous claim, when the path climbs the walls, it ends up on the other wall of the box and so the direction of the path on the top will be SE-NW and appear perpendicular (looking down) to the path on the bottom. This pattern continues. The paths on the bottom are always perpendicular to the paths on the top, establishing the claim.

**Claim 3**  *The path on the bottom always moves on pink squares. The path on the top always moves on pink squares if $c$ is odd; it always moves on white squares if $c$ is even.*

*Proof of Claim 3:* If $c$ were 0, the parity (pink/white) would change between bottom and top.

If $c$ were 1, and we’re not near a corner, the parity wouldn’t change.

This pattern continues for $c = 2, 3, 4, \ldots$. The only difficulty is that going around a corner changes whether or not the parity shifts. But by Claim 1, the
path always passes exactly two corners, removing the effect of corners on parity.

Claim

The proof of Proposition 4 now proceeds by cases.

Case 1: $a$ and $b$ even

We consider the subcases, $c$ even and $c$ odd, separately. If $c$ is even, we can divide the dimensions by two. The resulting box has the same properties as the original box and reduces to this or another case, but on a smaller box.

If $c$ is odd, consider the top of the box. On the top the path will travel on pink squares, going SE-NW and NW-SE.

There is no corner it can reach traveling this way, so the destination must be on the bottom. On the bottom the path travels on pink squares SW-NE and thus can only end at $110$. This is as the Proposition states ($c - a$ and $c - b$ are odd, hence equally even).

Case 2: One of $a$, $b$ is even, one is odd

Suppose, for example, that $a$ is even and $b$ is odd.

There is no corner on the bottom that the path can reach traveling SW-NE. If $c$ is even, the path will travel on white squares on top. The only corner possible is the SE corner, $101$, as predicted by the Proposition ($c - a$ is even, $c - b$ is odd).

If $c$ is odd, the path will travel on pink squares on top and can only end at the NW corner, $011$, and again agreeing with the Proposition.
Case 3: $a$ and $b$ odd

On the bottom, the path can end at the NE corner (110). If $c$ is even the path will travel on white squares on top and no corner is possible so the destination must be 110. This agrees with the proposition.

But if $c$ is odd, there will be three possible corners: 110 on the bottom, 011 and 101 on the top. Since all three dimensions are odd, let $a = 2k + 1$, $b = 2n + 1$, $c = 2m + 1$. To analyze the situation, we consider a smaller box, a box with dimensions $(k, n, m)$.

The paths on the two boxes are identical. On the smaller box one travels on the outside a distance of $m$ double blocks,

on the larger box one travels $2m + 1$

single blocks.

The difference is made up by the fact that the outer path is slowed by exactly two corners.
Thus, Case 3 reduces to the smaller box. Note that the relative evenness of 
\[ c - b = 2m + 1 - (2n + 1) = 2(m - n) \] and 
\[ c - a = 2m + 1 - (2k + 1) = 2(m - k) \]
is the same as the relative evenness of \( m - n \) and \( m - k \).

Since destinations are preserved when a box is magnified equally in all directions, we have:

**Corollary 1** For any rational \( a, b, c \) with \( a, b \leq c < a + b \), \( E(a, b, c) = \text{101, 011, or 110.} \)

## 4 Non-triangular Dimensions

If one dimension is greater than the sum of the other two, the smaller two dimensions can be subtracted from the largest. The relationship between the destinations of the two boxes is given by the following:

**Proposition 5** Let \( E(a, b, c) = xyz \). Then

- \( E(a + b + c, b, c) = x(y \oplus x)(z \oplus x) \),
- \( E(a, a + b + c, c) = (x \oplus y)y(z \oplus y) \) and
- \( E(a, b, a + b + c) = (x \oplus z)(y \oplus z)z \),

where \( \oplus \) is the ‘exclusive or’ operation (sometimes written XOR): \( 0 \oplus 0 = 1 \oplus 1 = 0, \ 0 \oplus 1 = 1 \oplus 0 = 1 \).

**Proof:** This proposition is the chief reason for the labels we placed on the corners, \textbf{000, 001, . . .}. If the first character is \textbf{0}, then the corner is in the YZ-plane (\( x \) coordinate is \( 0 \)). If the first character is \textbf{1}, then the corner is in the plane \( x = a \). Similarly, the second character registers whether the corner is in the plane \( y = 0 \) or the plane \( y = b \).
The effect of adding to one dimension the sum of the other two can be described in terms of the characters of the labels. Suppose, for example, \( a \) and \( c \) are added to \( b \).

The additional length rotates a path traversing the \( b \) direction exactly 180°.

If the destination on the small box is on the left, that is, if \( y = 0 \), then the path will travel the \( b \) length of the box an even number of times. Then the path will be rotated an even number of times so there will be no effect on the destination. In other words,

\[
E(a, a + b + c, c) = E(a, b, c)
\]

which coincides with the statement in the proposition, since \( xyz = (x \oplus y)y(z \oplus y) \) when \( y = 0 \).

On the other hand, if the destination on the small box is on the right, that is, \( y = b \) (\( y = 1 \)), then this has the effect of changing the ending corner’s other two coordinates. Again, this agrees with the statement,

\[
E(a, a + b + c, c) = (x \oplus y)y(z \oplus y).
\]
5 Infinite Paths

There are infinite paths.

**Proposition 6** If $r$, $s$, and $t$ are linearly independent as vectors over the field of rational numbers and $r, s < t < r + s$, then $L(r, s, t) = \infty$.

**Proof:**

Consider the unfolding of a box with a finite path.

The diagram is contained in a square. Each side of the square is a linear combination of $a$, $b$, and $c$. In the case above, the square is $3a + 2c$ wide and $3b + 2a$ high.
The sums of the coefficients in the linear expressions note the number of times each dimension of the box is traversed. For example, the sum of the coefficients of \(a\) is 5 and indeed the short side \(a\) is crossed five times in the path. The sums of the coefficients of \(a\) and \(b\) are odd; the sum of the coefficients of \(c\) is even and so the destination of the path is 110.

Now suppose for linearly independent \(r, s, t\), that \(L(r, s, t)\) is finite. Consider the unfolding. The sides of the square containing the unfolding will each be linear combinations of \(r, s,\) and \(t\); as sides of a square, they will be equal. Since \(r, s, t\) are linearly independent, we must have that the coefficients for the two sides are identical. That means that the sums of the two coefficients for each of \(r, s\) and \(t\) are even. Hence the destination of the path is 000.

**Definition 2** Call a triple \((a, b, c)\) balanced if in the unfolding, the coefficients of \(a, b,\) and \(c\) in the expressions for the lengths of the sides of the unfolding are the same.

**Claim 4** The set of points in the color triangle whose barycentric coordinates are balanced is open.

**Proof of Claim 4:** If \((a, b, c)\) is balanced, then sufficiently slight changes in \(a, b, c\) will not change the topology of the path in the unfolding. Imagine the effect on the unfolding of making a slight increase, say, in \(b\). Everything in the diagram moves. But the point at the upper right corner will move up and to the right at an angle of 45° because \((a, b, c)\) is balanced—every change to one side of the square is matched by an identical change to the other. The same is true for any (sufficiently slight) change in \(a\) or \(c\). Thus the destination in the altered system remains the same. Note that the altered path will cross the same edges and in the same order.

Using the claim, we can choose rationals \(r', s', t'\) approximating \(r, s\) and \(t\) which have an equivalent unfolding, that is, \(E(r', s', t') = 000\). But we will still have \(r', s' < t' < r' + s'\) and by the corollary to Proposition 4, \(E(r', s', t')\) must be 101, 011, or 110. This is a contradiction and the Proposition is proved.

### 6 Questions

1. The key region in the color triangle is the triangle in the middle, the set of triangular points. We can show ([4]) that on every straight line in the non-black sector of that triangle from the center to the edge consists of points of a single color.
We can prove, in addition, that points along that path represent infinite geodesics if and only if the point on the edge has rational barycentric coordinates.

In view of Proposition 5, this settles the question except that we still have no simple description of the answer. Is there a closed-form description of the destination of geodesics on boxes of a given dimension?

2. Calculating the length of a geodesic seems difficult.

We conjecture that for $a < b \leq c < a + b$:

$$L(a, b, c) = \frac{c^2 - a^2 - b^2 + ac + bc - ab}{d} - (a + b + c),$$

where $d$ is the greatest common divisor of $c - a$ and $c - b$.

Is this true, and is there a closed expression for $L(a, b, c)$ in general?

3. On a rectangle, the length of the bouncing path is equal to the least common multiple of the dimensions of the rectangle. The length of the geodesic on the box is in many ways analogous.

What properties does $L(a, b, c)$ share with the least common multiple of $a$, $b$, $c$?

4. Pursuing the analogy, on the rectangle, there are closed paths that do not hit corners if and only if the dimensions of the rectangle are relatively prime.

Under what circumstances are there loops (closed paths that do not hit corners) on a box with integral dimensions $(a, b, c)$?

We mean, of course, loops that strike every edge at integral distances from the ends of the edge. Note that there are boxes with closed paths where the dimensions are relatively prime, $(3,4,5)$, for example.
Is the absence of loops a property (for triples) analogous, in some way, to relative primality?

References