Squaring the Plane

Frederick V. Henle and James M. Henle

1. INTRODUCTION. This research was inspired by two lovely pieces of mathematics. The first is the discovery by William T. Tutte, A. H. Stone, R. L. Brooks, and C. A. B. Smith of squares with integral sides that can be tiled by smaller squares with integral sides, no two alike. Tutte tells the story in "Squaring the Square," a beautifully written article that conveys vividly the excitement of mathematical research [**9**]. It became widely-read in 1958 when it was reprinted in Martin Gardner's "Mathematical Games" column in *Scientific American*. It undoubtedly played a role in inspiring many to become mathematicians.

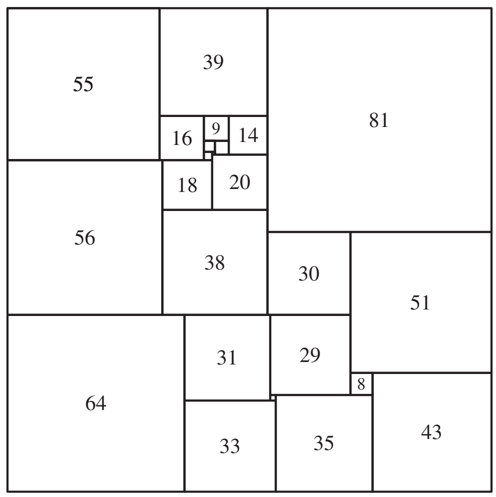

Figure 1. 175 × 175 perfect square.

The second piece is the well-known tiling of the plane by squares whose sides are the Fibonacci numbers (Figure 2).[1] The tiling is elegant, but in light of Tutte's work, possesses a minor flaw—it contains two squares of the same side. The flaw is easily remedied. We can use the squared square in Figure 1, together with a square of side 175 (Figure 3). This works, but some of the elegance is lost.

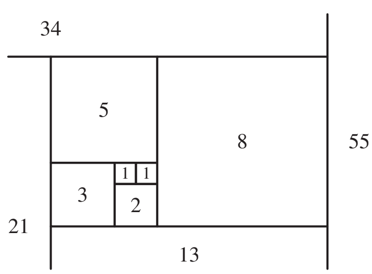

Figure 2. Fibonacci tiling.

[1]Coincidentally, this tiling appears in the same volume of Gardner's columns as "Squaring the Square" [**1**].

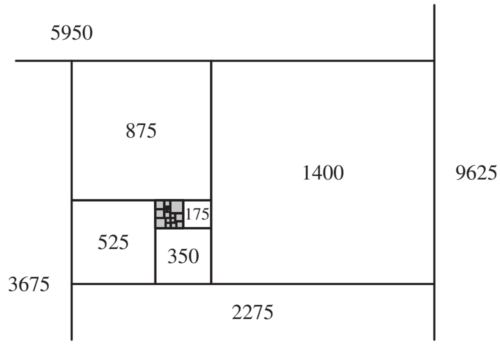

Figure 3. Fibonacci rectangle seeded with Tutte's perfect square.

In looking for a natural tiling that doesn't repeat squares, one quickly discovers that the first five squares fit together neatly (Figure 4) but it becomes progressively more difficult to add consecutive squares without overlapping or leaving gaps. This suggests the following question: Can the squares with whole-number sides, one of each size, be fitted together to tile the plane? The answer is that they can. In succeeding sections we will prove this, discuss the history of the problem, and pose a number of questions.

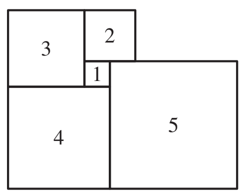

Figure 4. The first five squares.

2. SQUARING THE PLANE. We are going to show that the plane admits a tiling that uses exactly one square of side-length n ($n = 1, 2, \dots$) and uses it exactly once. Our approach focuses on rectangles and ells. By "ell," we mean any six-sided figure whose sides are parallel to the coordinate axes.

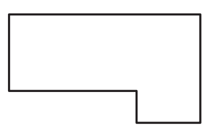

Figure 5. An ell.

The following extends Tutte's use of "perfect" for rectangles and squares.

Definition 1. A figure is *perfect* if it is composed entirely of squares of different sides.

All the remaining figures in this paper will be perfect. The key to our result is Lemma 5, which states that given any perfect ell, it is possible to add squares to it to form a perfect rectangle.

When we add squares to a perfect figure, keeping it perfect, we'll say we are "puffing it up." When we puff an ell up to form a perfect rectangle, we'll say we are "squaring up the ell." We can then "square the plane" as follows:

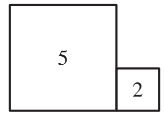

Figure 6. A perfect ell.

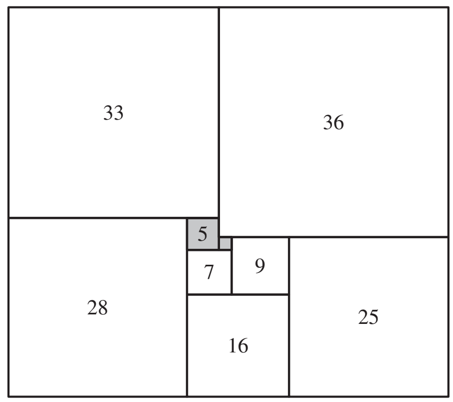

Figure 7. The ell of Figure 6 squared up.

1. Start with any perfect ell and square it up.
2. Create a new ell by attaching to the rectangle the smallest square not yet used.
3. Square this ell up, making sure that new squares are added in all four directions.
4. Repeat steps 2 and 3 *ad infinitum*.

Definition 2. An ell in *standard position* is an ell oriented so that the single reflex angle is at the upper right. We refer to the sides by the uppercase letters in Figure 8 and their lengths by the corresponding lower case letters.

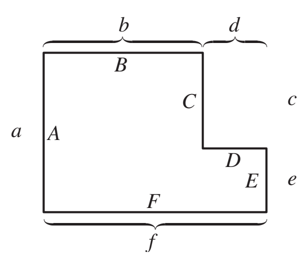

Figure 8. An ell in standard position.

When we add squares to a figure, we always do so in such a way that an added square's side-length matches that of at least one side of the figure against which it is placed. That means that the only squares we add to an ell in standard position are squares adjacent to sides A, B, C, D, E, or F of the appropriate lengths. We call these operations **A**, **B**, **C**, **D**, **E**, and **F**. So, for example, applying **B** then **A** to an ell produces a larger ell (Figure 9). We denote sequences of operations by sequences of letters with the understanding that they are applied from left to right (so the combination of operations in Figure 9 would be written **BA**).

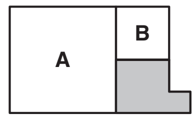

Figure 9. An application of **BA** to the ell of Figure 8.

Note that in any ell in standard position, the length of side A is the sum of the lengths of sides C and E,

$$a = c + e,$$

and the length of side F is the sum of the lengths of sides B and D,

$$f = b + d.$$

Thus, to describe the dimensions of such an ell, we need describe only the lengths of sides B, C, D, and E.

Definition 3. A 4-tuple $\langle b, c, d, e\rangle$ describes the dimensions of an ell if the sides B, C, D, and E of the ell are b, c, d, and e, respectively.

Definition 4. A side of an ell is *composite* if it is not the side of a single square in the ell.

In squaring up an ell, we make particular use of the operations **B**, **F**, and **ED**. For that reason, the following definition is most useful:

Definition 5. A perfect ell is *regular* if each of the moves **B**, **F**, and **ED** either results in a perfect ell in standard position or results in a perfect rectangle.

Figure 10 shows the smallest regular ell.

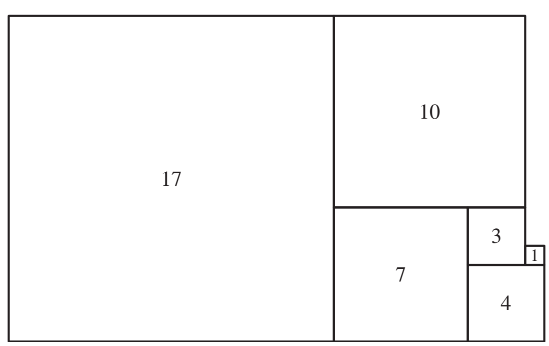

Figure 10. The smallest regular ell has dimensions $\langle 27, 12, 1, 5\rangle$.

If an ell with dimensions $\langle b, c, d, e\rangle$ is regular, then there can be no squares with sides b, f, e, or $d + e$ in the ell. In particular, sides B, F, and E must be composite.

In addition, it must be that $c \geq d + e$ (or else move **ED** would create an ell that is not in standard position). These properties—no squares of sides b, f, e, or $d + e$, and $c \geq d + e$—are both necessary and sufficient for regularity.

Lemma 1. *Every perfect ell P in standard position can be puffed up to form a regular ell without increasing the length of side D.*

Proof. Let the dimensions of P be $\langle b, c, d, e\rangle$. In every perfect ell in standard position, either A or F is the longest side, and whichever side is longest is necessarily composite. We may assume that $a \leq f$, since if not, then side A is the longest side and composite, so **A** can be performed, and in the new ell the length of side A is less than the length of side F (Figure 11).

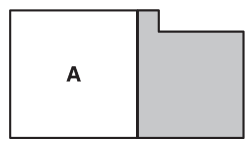

Figure 11. **A** makes $a \leq f$.

Since $a \leq f$, side F is the longest side and is composite, so **F** can be performed. This allows us to assume further that $c > d$. If not, we can perform **FABA**, and in the new ell, the length of side C is greater than the length of side D (Figure 12). Note that **FABA** does indeed puff up the ell—the first square, added by **F**, is larger than any in the ell, and each new square is larger than the one before. Note also that the length of side D is unaffected.

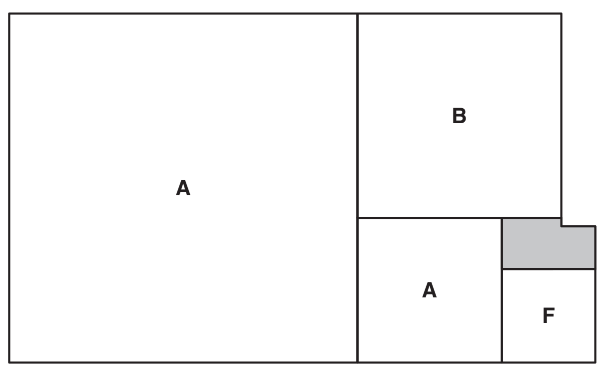

Figure 12. The operation **FABA**.

With our assumptions now ($f \geq a$, $c > d$), we perform **FABA** (Figure 13). Let $\langle b', c', d, e'\rangle$ be the dimensions of the ell formed after **FABA** (d, since **FABA** doesn't affect side D). It should be clear that performing either **F** or **B** to this new ell would result in a perfect ell in standard position.

To check move **ED**, note first that move **E** adds a square of side $e' = b + d + e$. This is larger than $b + d$ but smaller than $b + c + d + e$, hence a square not previously used. Applying **D** after **E** adds a square of side $d + e' = b + 2d + e$. This is larger than the square just added, but again less than $b + c + d + e$, since $c > d$. The result

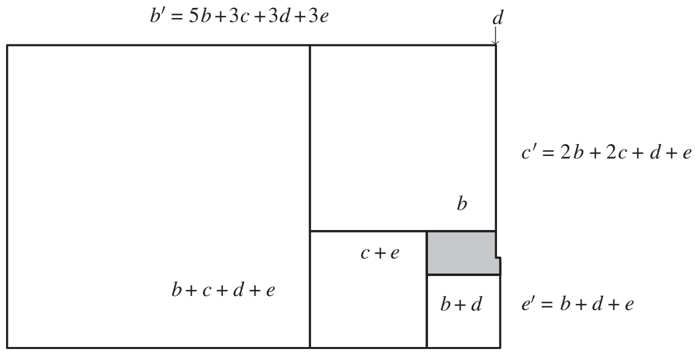

Figure 13. New dimensions after **FABA**.

then, is a perfect ell. To show that the ell is either a rectangle or in standard position, we need only the fact that $c' \geq d + e'$. But $c' = 2b + 2c + d + e$ and $e' = b + d + e$, and using $c > d$, we have $c' = b + 2c + e' > d + e'$. ∎

Key to proving our main lemma (Lemma 5) is an analysis of $c - e$ modulo d. For a regular ell $c \geq d + e$. Consequently, we can express c as $kd + e + i$ where $k \geq 1$ and $0 \leq i < d$.

Lemma 2. *If a regular ell has dimensions $\langle b, c, d, e\rangle$ such that $c = kd + e + i$, with $k \geq 2$ and $0 \leq i < d$, then the sequence of moves **BFA** produces a regular ell with dimensions $\langle b', c', d, e'\rangle$, where $c' = (k-1)d + e' + i$.*

Proof. Performing **BFA** (Figure 14) produces an ell with dimensions

$$\langle b', c', d, e'\rangle = \langle 3b + c + d + e, b + c, d, b + d + e\rangle.$$

Computing, we obtain

$$c' = b + c = b + kd + e + i = (k-1)d + b + d + e + i = (k-1)d + e' + i.$$

To see that the new ell is regular, first note that a move of **B** or **F** would result in a perfect ell in standard position.

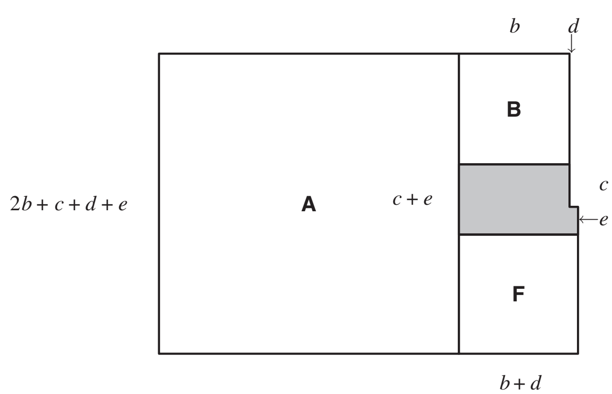

Figure 14. The operation **BFA**.

To see that **ED** would add only new squares, observe that **BFA** added squares of sides $f = b + d$, b, and $a' = 2b + c + d + e$. Move **ED** would add squares of sides $e' = b + d + e$ and $d + e' = b + 2d + e$. On the other hand, we have

$$2b + c + d + e > b + 2d + e > b + d + e > b + d > b,$$

the first inequality following from the fact that $c > d$.

All that remains to show that the ell is regular is that $c' \geq d + e'$. This follows from $c' = (k - 1)d + e' + i$, where $k \geq 2$. ∎

Lemma 3. *If a regular ell with dimensions $\langle b, c, d, e\rangle$ is such that $c = kd + e$ for some positive integer k, then the ell can be squared up.*

Proof. We can apply Lemma 2 until we have a regular ell whose dimensions $\langle b', c', d, e'\rangle$ satisfy $c' = d + e'$. This ell can then be squared up with **ED**. ∎

Lemma 4. *Every regular ell can be squared up.*

Proof. Given a regular ell with dimensions $\langle b, c, d, e\rangle$, we have $c = kd + e + i$ with $k \geq 1$ and $0 \leq i < d$. We do induction on d to show that the ell can be squared up.

If $d = 1$, then $i = 0$ and we are done by Lemma 3. Now assume that the result holds for any ell with dimensions $\langle b^*, c^*, d^*, e^*\rangle$ such that $d^* < d$. We apply Lemma 2 iteratively until we have an ell with dimensions $\langle b', c', d, e'\rangle$ with $c' = d + e' + i$. Then **ED** produces an ell with dimensions $\langle b', i, d + e', d + 2e'\rangle$ (Figure 15). By flipping this over we obtain in standard position an ell with dimensions $\langle 2e' + d, e' + d, i, b'\rangle$ (Figure 16).

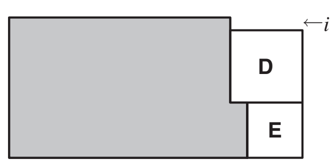

Figure 15. The operation **ED**.

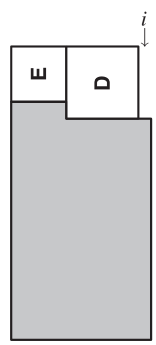

Figure 16. A flip after **ED**.

By Lemma 1 we can puff this up to a regular ell without increasing the length of side D (which is i). Since $0 \leq i < d$, by the inductive hypothesis this flipped ell can be squared up. Clearly, the unflipped ell can also be squared up, completing the proof of the lemma. ∎

Our main lemma follows immediately:

Lemma 5. *Every perfect ell can be squared up.*

Proof. Given an ell, we puff it up to a regular ell (Lemma 1), then square it up (Lemma 4). ∎

In other words, every perfect ell is a part of a perfect rectangle. We have now completed the spadework necessary to answer the tiling question raised in the introduction:

Theorem. *It is possible to tile the plane with nonoverlapping squares using exactly one square of each integral side.*

Proof. It is easy to see how to carry out the plan described earlier. We start with any perfect ell and grow it by adding squares as described in the algorithm whose steps we presented in Section 2.

Note that we don't move squares around; once added to the figure they are fixed. In particular, we never actually flip the figure as in the proof of Lemma 5—the flipping described there is simply to demonstrate that we can square up a particular ell. Note also that we are guaranteed to fill the plane since we are careful to grow the figure in all four directions in each cycle of the algorithm.

Finally, for any n the square of side n will be added to the figure by or before the nth time we perform step 2 of the algorithm. Since we perform step 2 infinitely many times, we are guaranteed to incorporate in the tiling a square of each integral side. ■

3. REFLECTIONS. In preparing this paper, we learned some of the history of the problem. The question was first posed by Solomon Golomb in a 1975 article in the *Journal of Recreational Mathematics*. He called it the "heterogeneous tiling conjecture" and challenged readers to prove or disprove it.

Martin Gardner wrote about the question four years later in his column in *Scientific American*, later anthologized in [**2**]. He described one approach to a solution, a fairly orderly tiling of roughly three-quarters of the plane with squares and reported that Verner Hoggatt Jr., then editor of *The Fibonacci Quarterly*, had shown that no square in the tiling appeared more than once.

Grünbaum and Shepard wrote about the problem in their 1987 book *Tilings and Patterns* [**4**]. They described there a second way in which a squared square S can generate a tiling of the plane (in addition to the method indicated in the present paper): Take a second copy of S and expand it to a square S_1 such that the smallest square in S_1 is the size of the original square S and fit S into that square. Take another copy of S and expand it to S_2 so that its smallest square is the size of S_1, and so on. Grünbaum and Shepard record the observation of Carl Pomerance that in every tiling of the plane by unique squares known at that time, the sides of the squares grow exponentially.

In 1997, Karl Scherer [**6**] succeeded in tiling the plane using multiple copies of squares of all integral sides. The number $t(n)$ of squares of side n is finite but not bounded. He describes his tiling as "size-alternating," in that no two squares of the same side share any portion of an edge (though they may share a corner).

4. QUESTIONS. Tiling is an enormous field. This theorem might be said to reside in the subfield that concerns infinite tilings of the plane that use exactly one specimen each of a well-defined collection of similar figures. Much work has been done here and many questions remain:

Efficiency. The algorithm presented in this paper is extravagant in that the ratio of the largest square used so far to the smallest square not yet used rapidly diverges. The procedure for squaring up, for example, when applied to the smallest possible ell, a 2×2 square next to a 1×1 square, ends in a rectangle with dimensions $110648136520515472 1693 \times 26596485578522 03795117$. The smallest square not used at this point is 4×4.

The squaring-up procedure can certainly be improved. By way of illustration, take the ell in Figure 6. This can be squared up to a 69×61 rectangle with the sequence of operations **FEFEABC** (Figure 7). Our procedure, however, doesn't square the ell up until a rectangle is reached with dimensions approximately $(5.0 \times 10^{14272}) \times (5.8 \times 10^{14272})$.

Can our squaring-up procedure be improved in some well-defined way? Does an algorithm exist for tiling the plane that methodically expands a connected island of squares in such a way that the ratio of the largest square used to the smallest not yet used is bounded by a polynomial?

Simple tilings. A perfect figure is *simple* if it contains no perfect subrectangle. Our tiling is far from simple. Is there a simple tiling of the plane using one specimen of each integral side?

The half-plane and quarter plane. Can the half-plane be squared? Can the quarter-plane be squared? We are especially interested in this question because if it is possible to tile a quarter plane four times using, altogether, every integral square just once, then it's possible to tile the plane using all the integral squares plus one square of any given side (say π).

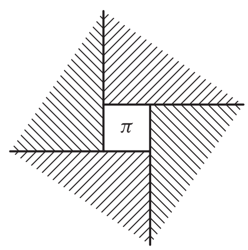

Figure 17. $\pi \times \pi$.

Rational squares. The algorithm of our theorem can easily be used to tile the plane using one square of every rational side. As with integral squares, we don't know if a similar procedure works for the half- or quarter-plane.

Positive and negative squares. We can tile the plane with squares whose sides are natural numbers and with squares whose sides are rationals. What about squares whose sides are (positive and negative) integers? We interpret the effect of placing a small negative square on top of a large positive one as removing a part of the large square. Once again, our algorithm works easily for this. Just as placing a positive square next to a rectangle creates an ell, so does placing a negative square on a corner of a rectangle. With care, no point of the plane will be touching more than three squares (one negative, two positive).

Odd squares. Can the plane be squared with all the odd squares? This seems unlikely to us. Can the plane be squared with *some* of the odd squares? In general, what well-defined subsets of the natural numbers can be used to tile the plane?

Coloring. Neither our tiling nor the Fibonacci tiling can be 3-colored. Is there a 3-colorable tiling of the plane using exactly one square of each integral side? Is there a simple algorithm for 4-coloring the tiling described in this paper?

Space. Can space be cubed?

Triangles. Scherer proved that the plane cannot be tiled with equilateral triangles of different sizes if one triangle is smallest [**7**]. He has found a way of tiling the plane, however, with different sizes of iscoceles right triangles and with enlargements of certain other triangles [**8**]. Pomerance proved that it is possible to tile the plane with one rational triangle of each congruence class such that any two neighboring triangles share either an entire side or just a vertex [**5**]. Left open is the question: Can the plane be tiled with all rational equilateral triangles so that no triangle has an infinite number of neighbors?

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