Chapter 2

Interacting Populations.

2.1 Predator/ Prey models

Suppose we have an island where some rabbits and foxes live. Left alone the rabbits have a growth rate of 10 per 100 per month. Unfortunately for the rabbits, the foxes find them to be most tasty! The number of rabbits eaten by foxes depends on how many foxes there are and how easy it is for them to catch rabbits - i.e., how prevalent the rabbits are. Foxes would die off if there were no rabbits around to eat, so their "growth rate" if there are no rabbits is $F' = -.04F$. Let $R(t)$, and $F(t)$ stand for the number of rabbits and foxes in month t. We get the following *system of differential equations*.

$$R' = .1R - .000084RF$$

$$F' = .00004RF - .04F$$

If $R(0) = 2000$ and $F(0) = 600$ what can we say about the populations of rabbits and foxes in the coming months?

The methods we have seen can be used to handle several functions at the same time. In particular, if we have the *initial value problem* where we know $R'(t)$ and $F'(t)$ and initial values of $R(0)$, and $F(0)$ then we can use Euler's method to find future values of R and F. The process works almost exactly as before. We use the current values of the variables and the derivatives to approximate the next month's values. Specifically,

$$R(t + \Delta t) \approx R(t) + R'(t)\Delta t$$

$$F(t + \Delta t) \approx F(t) + F'(t)\Delta t$$

See if you can fill in the following table. For simplicity of calculation we have chosen $\Delta t = 1$.

t	$R(t)$	$F(t)$	$R'(t) = .1R - .000084RF$	$F'(t) = .00004RF - .04F$
0	2000	600	99	24
1	2099	624	100	27
2	2199	652	99	31
3				
4				
5				
6				
7				
8				

How do we need to modify the program EULER1.BAS so that it will solve the rabbits and foxes problem? Make the changes given below and then run the program to get the values for the above table.

```
EULER1.BAS(revised):
tinitial = 0
tfinal = 20
numberofsteps = 20
deltat = (tfinal - tinitial)/numberofsteps
r = 2000
f = 600
FOR k = 1 to numberofsteps
    rprime = .1*r - .000084* r*f
    fprime = .00004*r*f - .04 * f
    r = r + rprime * deltat
    f = f + fprime * deltat
    t = t + deltat
    PRINT "t=" t "r=" r "f="f
NEXT k
```

Now try this problem using SLINKY. Let the program run for awhile, at least until $t = 150$ and see what happens. What do you learn about the interaction of these species? Do both populations reach a certain level and then stay at that level?

Try different initial populations of foxes: $F = 1200$ and then also try $F = 3000$. What happens in these cases? Is the interaction the same in all cases? What if we start with extremely few foxes (40, 20)? What if we start with extremely many foxes (4000, 8000)?

(i)Do you get cyclic changes in every case? (ii) When you do what are the highest and lowest levels of the populations? (iii) How long does it take for the periodic behavior to start? (iv) What is the length of the cycle?

2.2 Systems of Differential Equations and Periodicity

Before looking further at systems which exhibit periodic behavior we first review some terms from the study of trig functions. Roughly speaking, a function is periodic if it is cyclical, the same values repeat over and over. Examples from every day life include, the length of a day which has a yearly cycle or the phases of the moon which has a monthly cycle.

A function $f(x)$ is *periodic* if there is a positive number T for which

$$f(x+T) = f(x)$$

for all x. The smallest such T is called the *period* of $f(x)$. The *frequency* of a periodic function is how often the cycle repeats or the number of cycles per unit time. So the phases of the moon has frequency once per 28 days. In general if T is the period, then $\frac{1}{T}$ is the frequency.

Graph the sine and cosine functions and verify that both have period 2π. Both these functions attain a max height of 1 unit, this is called the *amplitude* of these functions. Look at the function $a \sin bt$ and for various values of a, b and observe what effect these values have on the period and amplitude of the function. Repeat for $a \cos bt$.

Equilibria An equilibrium point for a system of differential equations is a set of initial conditions for which the current rates of change will be zero. In other words, once the system is at the equilibrium point it will never change. In the homework you will look at some different types of equilibria for a system of differential equations.

2.3 Problems for Chapter 2

Exercise 2.1. Each sentence describes the rate of change for Plingos (P's) in terms of the current population of Plingos and Quiepes (Q's). In each case match the statements with an appropriate equation.

(i) The Plingos disappear faster when there are lots of Quiepes.

(ii) The Plingos increase, the more Quiepes there are.

(iii) Left on there own, the Plingos would die out.

The equations (assume a, b, c, d are positive constants): (a) $P' = -aP$, (b) $P' = -bQ$ (c) $P' = cQ$ (d) $P' = dP$

Exercise 2.2. Assume that on a desert island a small rabbit population grows at a rate of 15% per month. Assume that the population growth is logistic using the logistic model we developed in the last assignment and in class. The carrying capacity of the island is 10000 rabbits. If at time $t = 0$ there are 4200 rabbits, roughly how long will it take the population to stabilize?

Exercise 2.3. Suppose in addition to the rabbits there are foxes on the island. These foxes eat rabbits, and would die off if there were no rabbits to eat. A reasonable formula for the growth rate of the fox population depends on $R(t)$ the current number of rabbits:

$$F'(t) = .00004 * F(t) * R(t) - .04F(t)$$

The foxes eat rabbits at a rate of $.0005 * R(t) * F(t)$, so a new equation for $R'(t)$ will include that as a term (and also a term as in 1.).

Assume that the rabbits are at their stable population when 100 foxes arrive on the island. Sketch these two populations over time on the same axes.

Now look at the same model starting with 20 foxes. Starting with 500 foxes? Is it possible to start with enough foxes so that the rabbit population eventually dies out? Explain.

Write a paragraph explaining your conclusions.

Exercise 2.4. In this problem we analyze the interaction of two species which compete with each other for scarce resources (perhaps food or territory). The objective of this problem is to see if one population will eventually drive out the other, or if both will survive and share resources.

$$x' = .15(1 - .005x - .01y)x$$

$$y' = .03(1 - .006x - .005y)y$$

(a) If x and y are both small then the parenthetical terms are both approximately 1. Which species exhibits a faster growth rate in this circumstance?

(b) Modify `Euler1.bas` so that it finds the populations in this problem at $t = 10$, starting with $x(0) = 150$ and $y(0) = 25$

(c) Examine this system with a variety of different starting conditions. Start with $(x, y) = (10, 10)$; $(150, 25)$; $(300, 10)$; $(200, 200)$, $(50, 200)$. You will probably want too more as well Do all the starting conditions lead to the same *eventual* outcome or are there various outcomes? YOU DO NOT NEED TO PRINT OUT ALL YOUR GRAPHS!

(c) The *principle of competitive exclusion* in ecology states that you cannot have a stable situation in which two species compete for the same resource - one will eventually crowd out the other. Is the mathematical model of this problem consistent with this principle?

Exercise 2.5. Predator Prey Models, Equilibria and Periodicity. In this problem we continue our analysis of the interaction between the population of a predator y and its prey x. The model we use here is called the May model.

$$x' = .6x(1 - \frac{x}{10}) - \frac{.5xy}{x+1}$$
$$y' = .1y(1 - \frac{y}{2x})$$

(a) Show that the constant functions $x(t) = 10$ and $y(t) = 0$ are a solution to the equations. Explain the behavior of this equilibrium solution.

(b) Is $x(t) = 0$, $y(t) = 0$ and equilibrium solution as well?

(c) Is $x(t) = \frac{-23\pm\sqrt{889}}{6}$, and $y(t) = \frac{-23\pm\sqrt{889}}{3}$ an equilibrium solution?

(d) Graph the solution to the May model using the initial conditions $x(0) = 1.13$ and $y(0) = 5$. (Use a computer program!) These initial conditions are very close to the equilibrium solution of part (c). Does this suggest that the solution of (c) was a *stable* equilibrium or an *unstable* one.

(e) Change the initial conditions to $x(0) = 5$ and $y(0) = 5$. Graph this and compare with the examples we did in class ($x = 10$, $y = 1$; $x = 8$, $y = 2$). In particular, compare the shapes of the graphs, periods, amplitudes, and time interval between the peak of y and the peak of x.

(f) Finally, suppose that the prey species x is actually an agricultural pest, while the predator y does not harm crops. Farmers would like to eliminate the pest and propose to do so by bringing in large numbers of the predator. Does this strategy work according to the May model? Suppose we start with a relatively large number of predators. $x = 5$ and $y = 50$. What happens? Does the pest disappear?

Chapter 3

Models of Linear Springs

3.1

Next we turn to a model of a linear spring. Imagine the spring hanging from the ceiling with a weight of mass m attached to it. We we look at how the displacement x and velocity v of the spring vary over time. The simplest model is to assume a linear relation between the position of the weight and the force exerted on the spring. So we say force $=-cx$ where the constant c will depend on the stretchability of our specific spring. Newton's law says that force = mass times acceleration = $m\frac{dv}{dt}$. Combining these two equations we get $mv' = -cx$. Or, $v' = -b^2 x$ where the constant $b = \sqrt{\frac{c}{m}}$. The choice of the name b^2 for the constant may seem odd, but note that it is in units of centimeters per second squared.

Using the above, and the fact that velocity is the derivative of position, we get a system of differential equations for a linear spring.

$$x' = v$$

$$v' = -b^2 x$$

To begin our analysis of this situation, start with the constant $b = 5$. We will initially consider the case where the spring being held at a position other than $x = 0$ and is motionless. That is, use initial conditions $x(0) = a$ and $v(0) = 0$. Try several values of a such as $a = 3, 5, -3, 10, -10$ and see what happens. How do the different values of a effect the position and velocity over time? Use the following table to keep track of your results.

b	a	pd of x	amp of x	pd. of v	amp of v
5	3				
5	5				
5	-3				
5	-5				
5	10				
6					
10					
8					

Next try different values of b for a given initial condition $x(0) = 5$. What happens now?

3.2 An exact solution

We return to our study of sine and cosine.

If $x = a \sin bt$ calculate x' and x'' (the second derivative).

$$x' =$$

$$x'' =$$

Repeat if $x = a \cos bt$.

$$x' =$$

$$x'' =$$

In both of these cases $x'' = -b^2 x$. Thus $x = a \sin bt$ and $x = a \cos bt$ are both possible solutions to the system $x' = v$, $v' = -b^2 x$. To find out which one fits precisely, we need to use the initial conditions.

A little work shows that $x(t) = a \cos bt$ is the right equation when we start with $x(0) = a$ and $v(0) = x'(0) = 0$. Thus the amplitude of this spring's position is precisely a and the frequency is precisely $\frac{b}{2\pi}$.

3.3 Equilibria

If we give the spring initial conditions $x(0) = 0$ and $v(0) = 0$, the spring will not move at all since then $x'(0) = v'(0) = 0$. In fact, the *equation* $x(t) = 0$ satisfies the system of differential equations for the spring.

3.4 Problems for Chapter 3

Exercise 3.1. Previously we assumed that $v(0) = 0$. Now we will consider what happens when an initial velocity is assumed.

$$x' = v$$

$$v' = -b^2 x$$

with initial conditions $x(0) = a$ and $v(0) = p$. Assume $b = 5$ per second, $a = 0$ and $p = 20$ cm/sec.

(i) Use Slinky to draw a graph of the motion of the spring.

(ii) Use your graph to estimate the period and amplitude of this system.

(iii) Find a formula for this system using the graph as a guide.

(iv) From your formula, determine period and amplitude. Do these depend on p and or b?

Exercise 3.2. Now take $a = 4$, $b = 5$ and give the spring an initial downward impulse of $p = -20$ cm./sec. Graph this system, give period and amplitude and compare with the case where $v(0) = 0$.

Exercise 3.3. What happens with other initial velocities (p)?