Chapter 6

Functions of Several Variables

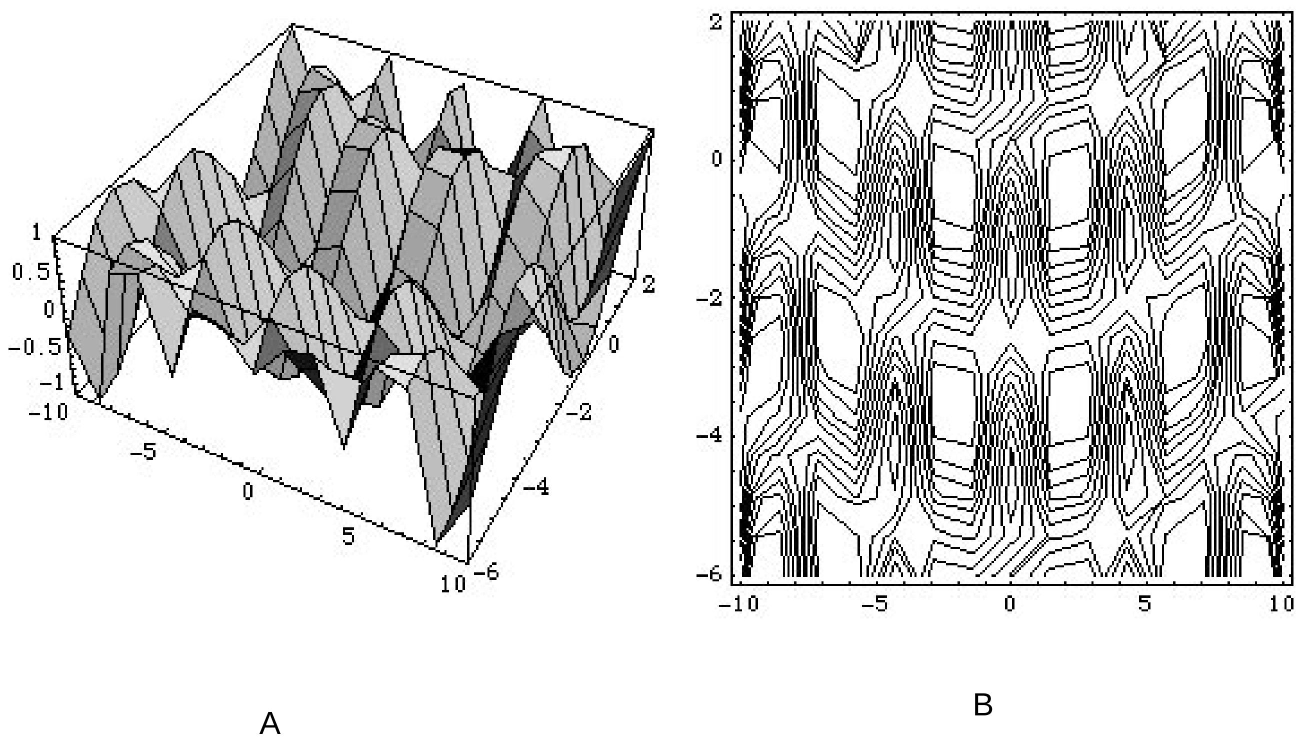

Figure 6.1: (A)3-d graph and (B) contour plot of $\sin x^2 + y$.

In the last section of this course we will begin to study functions of several variables. If you continue to Calculus III, this will be a major topic of the course.

6.1 Visualizing functions of 2 variables

One problem with thinking about functions of several variables is that they can be harder to picture than functions of just one variable. We will examine three ways to look at functions of 2 variables.

Contour Plots A *contour plot* is a two dimensional representation of a function of two variables. Weather maps that show temperatures are excellent examples, as are topographic maps that show the elevation of

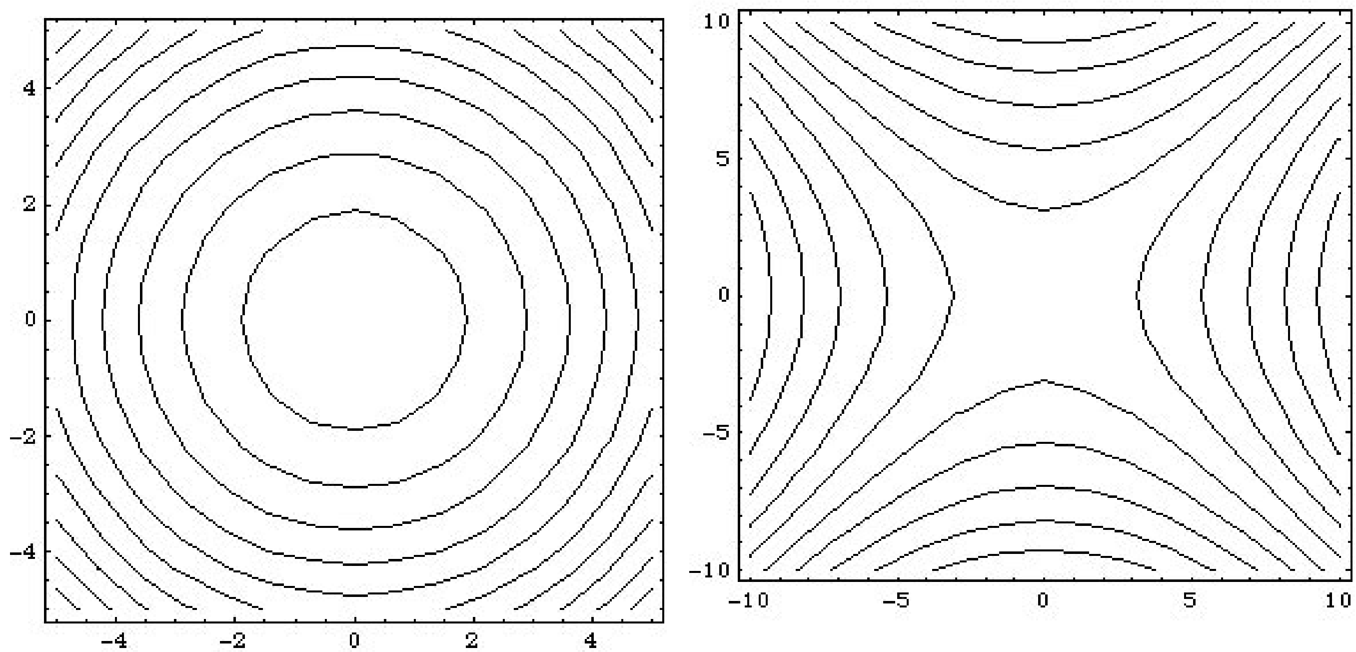

Figure 6.2: Max or Min point and a Saddle point.

land. A contour plot for a function $z = f(x, y)$, shows the (x, y) plane, which is the domain of the function. Lines are drawn on this plane indicating values of z, the function output. Each line represents a specific z value. For example, on a weather map, the x and y values give the location of a place, and $z = f(x, y)$ is the temperature in that place. The lines we see drawn on a weather map (called isobars) will show all the places where the temperature is exactly 60, 70, 80 etc.

By looking at a contour map of a function you should be able to tell where the function is changing slowly and quickly, where it is a maximum or minimum and where it stays constant. To stay constant you go along the contour lines, while to change the fastest you want to cross the contour lines as quickly as possible -that is, you should walk perpendicular to the contour lines.

We can use the program `Mathematica` to draw contour plots. The command for this is `ContourPlot[`$f(x, y), \{x, x_l, x_m\}, \{y, y_l, y_m\}$`]`, where $f(x, y)$ is the function you want to graph, and x_l, x_m and y_l, y_m give the domain of the x and y values respectively.

3-Dimensional Graphs Another way to look at a function of 2 variables is with a 3-d plot. These can be hard to draw. Luckily, the computer will draw them for us. To do this we will use the program `Mathematica`. The command for this is `Plot3D[`$f(x, y), \{x, x_l, x_m\}, \{y, y_l, y_m\}$`]`, where $f(x, y)$ is the function you want to graph, and x_l, x_m and y_l, y_m give the domain of the x and y values respectively.

Slices We can learn about a function of two variables by studying how the function responds to changes in just one variable. For example consider the function $f(x, y) = x^2y + 5y$. If we hold x constant at $x = 1$ then the function looks like $f(1, y) = 6y$. Holding x constant at any value and varying y we get a slice of this function in the y direction looks like a line. On the other hand, if we hold y constant and vary x we get a quadratic. For example, $f(x, 1) = x^2 + 5$.

What we are doing here is looking at a "slice" of the 3-d picture. In one case, we slice along the $x = 1$ plane. In the other case along the $y = 1$ plane.

6.2 Position, direction and distance

We will need to develop a notation for describing position and direction of points in the domain of a function of 2 variables. The domain is the (x, y) plane. Consequently, points in the domain are indicated by there (x, y) values. We will often be concerned with what happens to a function as we move from one point in the domain to another. We might indicate this by naming the point we are moving toward. Eg., If we are at the point $(-6, 4)$ we can move toward the point $(8, 3)$.

Alternatively we give the comparative amounts we are heading in the x and y directions. For example, if we say we are heading in the $(3, -4)$ direction, that means for every 3 units in the x direction we go, we go -4 units in the y direction. For example, if we are at the point $(2, 5)$ and are heading toward the point $(8, 1)$ then we are heading in the $(6, -4)$ direction.

The distance between two points is determined by the length of the line segment between them. Using trigonometry we see that the distance between (a, b) and (c, d) is given by $\sqrt{(a-c)^2 + (b-d)^2}$.

practice

Fill in the blanks.
If you are at the point $(2, -1)$ and go 5 units in the $(3, 2)$ direction, then you will be at the point $(\ ,\ )$.
If you are at the point $(2, -1)$ and go $(\ )$ units in the $(1, 1)$ direction, then you will be at the point $(5, 2)$.
If you are at the point $(2, -1)$ and go 1 unit in the $(\ ,\ )$ direction, then you will be at the point $(3, -1)$.

6.3 Planes

A plane is a flat 2-dimensional surface. Just as any 2 points determine a unique line, 3 points determine a plane. A plane in 3-d has an equation of the form

$$ax + by + c = z$$

The coefficients a and b can be thought of as slopes, $a = \frac{\Delta z}{\Delta x}$, and $b = \frac{\Delta z}{\Delta y}$. That is, the coefficients a and b are the rate of change of z in the x and y directions respectively.

How do we find the equation for a plane? Consider the plane through the points $(0, 1, 2)$, $(0, 2, 5)$, $(1, 1, 0)$. Notice that the first two points have the same x value and y changes by 1, thus $b = \frac{\Delta z}{\Delta y} = \frac{(5-2)}{(1-0)} = 3$. Similarly, since the first and third points have the same y value we can use them to find the x coefficient. $a = \frac{\Delta z}{\Delta x} = \frac{(0-2)}{(1-0)} = -2$. To find c, just plug one of the points into the equation: $c = z - ax - by$. Using the first point we get $c = 2 - (-2)0 - 3(1) = -1$. Check that all three points now lie on the line $(-2)x + 3y + (-1) = z$

Not every set of three points so easily leads to an equation for a line. We were lucky in this case, to have 2 points with the same x value and 2 points with the same y value.

Consider the plane $2x - y + 7 = z$. Draw (or have `Mathematica` draw) a contour map, a 3-d plot and slices for this. Notice that a change of one unit in the x direction (leaving y fixed) results in a 2 unit increase in z. What is a result of a one unit increase in y, leaving x fixed? Since a plane is flat, the result of changing one unit in the x direction is the same no matter where we make the change.

This is true if you go in any direction. For example, if you go 3 units in the x direction and 1 unit in the y direction z will always increase by $2 * 3 - 1 = 5$. Thus in the direction $(3, 1)$ the rate of change is always $\frac{5}{(\sqrt{(3^2+1^2)})} = \frac{5}{\sqrt{10}}$.

This fact is very important. On any plane, no matter where you are, the rate of change when you move in a given direction is the same.

6.4 Calculus

We have just looked at rates of change of functions that are planes. Really, we were looking at the derivative of the function. In a function of several variables $f(x, y)$, we define several *partial derivatives*. There will be one partial derivative to describe the rate of change of the function as each single variable changes holding all other variables constant. For example the function $f(x, y) = x^2y + 5y$ has $\frac{\partial f}{\partial x} = 2xy$, and $\frac{\partial f}{\partial y} = x^2 + 5$. We can take higher order partial derivatives as well. Continuing with our example, $\frac{\partial^2 f}{\partial x^2} = 2y$, $\frac{\partial^2 f}{\partial y^2} = 0$, $\frac{\partial^2 f}{\partial y \partial x} = 2x$ and $\frac{\partial^2 f}{\partial x \partial y} = 2x$.

The partial derivatives tell us the rate of change in the x or $(1, 0)$ direction and the y or $(0, 1)$ direction. To get the rate of change in other directions we just combine the derivatives. E.g., the rate of change in the $(3, 7)$ direction is {total change} /{total distance} = $\left(3\frac{\partial f}{\partial x} + 7\frac{\partial f}{\partial y}\right)/\sqrt{3^2 + 7^2}$.

The **tangent plane** of a function $f(x, y)$ at a point (x_o, y_o) serves much the same purpose as the tangent line of a function of one variable. It is a linear approximation of the function at that point and has the same first derivatives. It is easiest to see the tangent plane in point, slope, slope form

$$(z - z_o) = r(x - x_o) + s(y - y_o)$$

where $r = \frac{\partial f}{\partial x}$ and $s = \frac{\partial f}{\partial y}$, evaluated at the point (x_o, y_o).

For example, we find the tangent plane for $f(x, y) = x^2y + 5y$ at the point $(x_o, y_o) = (3, -2)$. This plane goes through the point $(x_o, y_o, z_o) = (3, -2, -28)$ and has $r = \frac{\partial f}{\partial x} = 2xy = -12$, $s = \frac{\partial f}{\partial y} = x^2 + 5 = 14$. So the tangent plane at this point is $(z + 28) = -12(x - 3) + 14(y + 2)$. We can rewrite this as $z = -12x + 14y + 36$.

The **gradient**, ∇ of a function at a given point is a direction. By definition it is $\nabla = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. The gradient gives the direction of steepest ascent of the function. Also, the direction of steepest descent is given by $-\nabla$. The rate of steepest ascent or descent will be the length of the gradient vector, that is $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$.

For example, consider again the plane $2x - y + 7 = z$. It has $\nabla = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2, -1)$. If we draw the contour plot of this plane, we would see that the direction in which we increase the fastest is indeed the $(2, -1)$ direction. The rate of steepest ascent will be $\sqrt{2^2 + (-1)^2} = \sqrt{5}$.

In general, if the function is not a plane, the direction of steepest ascent and descent will depend on the point we are at.

6.5 Problems for Chapter 6

Exercise 6.1. For each of the functions below do the following:

(a) Obtain a graph of the function (use Mathematica).

(b) Obtain a contour plot of the function (use Mathematica).

(c) Find all minimums, maximums and saddle points in the given domains (from the plots).

(d) Sketch 3 representative slices in the x direction and in the y direction.

<u>The functions:</u>

(i) $z = \sin(x)\cos(y)$ domain: $0 \le x, y \le 2\pi$;

(ii) $z = 2x + 4x^2 - x^4 - y^2$ domain: $-2 \le x \le 2, -4 \le y \le 4$

Exercise 6.2. Find the equation for the plane that goes through the points

(a) $(0, 0, 0)$, $(0, 1, 1)$ and $(1, 0, 1)$.

(b) $(0, 0, 99)$, $(0, 1, 99)$ and $(-12, 5, 99)$.

(c) $(0, 0, 0)$, $(1, 0, 4)$ and $(0, 1, 3)$

Exercise 6.3. For each function below find all partial derivatives of the first and second order, i.e., find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$

(a) $f(x, y) = 3x^2y^4 + 7x$

(b) $f(x, y) = e^{xy}$

(c) $f(x, y) = e^{x+y}$

(d) $f(x, y) = \sin(x)\cos(y)$

Exercise 6.4. (a) Show that if $f(x, y) = x^n y^m$ then the two mixed second order partial derivatives are equal. That is, show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

(b) Use your result from part (a) to show that the mixed partials of a polynomial are necessarily equal. What do you think is true in general?

Exercise 6.5. Find the tangent plane of the following functions at the following points.

(a) $f(x, y) = 3x^2 - 2xy + 12$ at $(-1, 0)$ and also at $(1, 2)$.

(b) $f(x, y) = \sin(x)\cos(y)$ at $(\frac{\pi}{4}, 0)$.

(c) $f(x, y) = x^3 + 4x^2y - 7xy^2 + 2xy - 5y + 11x + 13$ at $(-1, 1)$

Exercise 6.6. Find the gradient ∇ of the functions at the indicated points in problem 5.

Exercise 6.7. Suppose you are hiking on a mountain range which is in the shape of the surface $z = f(x, y) = x^2y - 2xy + 4x + 2y$. Suppose you are standing at the (x, y) coordinates $(1, 0)$.

(a) What is the height (z value) of your current position?

(b) What is the rate of change in your height if you walk parallel to the x-axis? the y-axis?

(c) What is the rate of change in your height if you walk toward the point $(2, 1)$? What about in the $(2, 1)$ direction?

(d) In what direction should you walk to go up the fastest? to go down the fastest? To remain level?

(e) Repeat parts (a) through (d) starting at the (x, y) coordinates $(0, -2)$.

(f) Can you find any extrema (maxima, minima or saddle points) for this function?

Exercise 6.8. Do parts (a) through (d) of the previous problem for the function $z = f(x, y) = xe^y - 5x^3y$ at the point $(1, 0)$.