

# Chapter 1

## Euler's Method, Modeling and the Computer

### 1.1 Lines, Functions, Tangents

One quantity  $y$  is a **function** of another  $x$ , if each value of  $x$  has a unique value of  $y$  associated with it. We will write  $y = f(x)$ . For example, expected high temperature is a function of the day of the year. Some functions are can be expressed by formulas, though not all can. Think of 2 functions which have no nice formulas. Think of a couple of functions you know that have nice formulas.

A **linear function** has the form  $y = f(x) = mx + b$ . Its graph is a (straight) line such that  $b$  is the vertical intercept, or value of  $y$  when  $x = 0$ , and  $m$  is the slope. The slope of a line is the rate of change of  $y$  with respect to  $x$ . I.e, if you increase  $x$  by one unit, then  $y = f(x)$  will increase by  $m$ . Check it!

Other functions have rates of change too. But they will not be the same for all  $x$  values. In fact, one of the things that makes a line special is its constant rate of change. The instantaneous rate of change of the function  $f(x)$  at the point  $a$  is the called the derivative of  $f$  at  $a$ .

**Tangent lines can be used to approximate functions** The *tangent line* of a function  $f(x)$  at a particular point  $x = a$  is a line which has the same  $y$  value as  $f(x)$  at the point  $a$  and the same rate of change as  $f(x)$  at  $a$ . If you graph the function  $f(x)$  and its tangent line at the point  $a$  and then zoom in near the point  $a$  you will see that the function and the tangent line look almost identical. Use the program GRAPH under the calculus menus to try this. The tangent line is a close approximation of the function near the point of tangency. This is a central idea in the work that we will be doing this term.

**Examples** (i) Louise is driving down Interstate 91. She notices that she is currently passing milepost 20 and going 90 miles an hour. Where will she be in 1 minute? (ii) The temperature of a cup of coffee is 130 degrees. You drop in an ice cube and the coffee starts getting colder at a rate of 10 degrees a minute. What is the temperature of the coffee in 30 seconds? What about in 20 minutes? Really?

**Moral:** If you know  $f(a)$  and  $f'(a)$  you can use these to approximate values of  $f(x)$  for values of  $x$  near  $a$ .

If  $f(0) = 10$  and  $f'(0) = 2$  then we can surmise that for small values of  $h$ ,  $f(0 + h) \approx 10 + 2h$ .

### 1.2 Euler's Method

This is a method for approximating a function from its derivative. It is based on the idea we have been discussing. If we know the derivative of a function at all points we can use this to approximate the function itself.

Suppose an apple is dropped off a bridge 100 feet above the water. The force of gravity will accelerate the apple so that its velocity is given by the formula  $f'(t) = -32t$ . How can we use this information to find out the position  $f(t)$  of the apple at any given time  $t$ ?

What happens after two seconds? Using the formula we saw above  $f(h) \approx f(0) + f'(0) * h$ , or in this case  $f(2) \approx 100 + 0 * 2 = 100$ . That is, the apple doesn't move at all in the first two seconds! That doesn't make a lot of sense! This approximation assumes that the velocity was constant throughout the whole first two seconds when in fact we know the velocity was changing throughout that time. In particular, the velocity at  $t = 1$  is given by  $v(1) = -32$ . So we can make a better approximation as follows:

(Position at  $t = 2$ )  $\approx$

(position at  $t = 1$ ) + (distance moved between  $t = 1$  and  $t = 2$ ).

Written as a formula, this is:

$$f(2) \approx f(1) + f'(1) * 1$$

$f(1)$  can be approximated by  $f(1) \approx f(0) + f'(0) * 1$ . So we get that  $f(1) \approx 100$  and  $f(2) \approx 100 - 32 = 68$

What we have done is broken up our interval from  $t = 0$  to  $t = 2$  into two steps and assumed different constant velocities for each. We can do even better by breaking our interval into 3 steps, 4 steps, 10 steps, etc. Let's see how this will work:

**Example:** Breaking the interval into 4 steps. In this case we will calculate the position at  $t = .5, 1, 1.5, 2$ , starting with our knowledge that  $f(0) = 100$ .

$$\begin{aligned} f(.5) &\approx f(0) + f'(0) * .5 &= 100 + 0 * .5 &= 100 \\ f(1) &\approx f(.5) + f'(.5) * .5 &= 100 - 16 * .5 &= 92 \\ f(1.5) &\approx f(1) + f'(1) * .5 &= 92 - 32 * .5 &= 76 \\ f(2) &\approx f(1.5) + f'(1.5) * .5 &= 76 - 48 * .5 &= 52 \end{aligned}$$

Before we learned that the tangent line approximation is best for values very near our original value. Thus we want each step we take in this process to be pretty small - that means we need to take a lot of little steps as opposed just a few big ones. This process of calculating  $f(t)$  from its derivative  $f'(t)$  and an initial value  $f(0)$  is called **Euler's method** after the 17th century Swiss mathematician. There are other methods for solving the *Initial Value Problem* you may learn them in subsequent calculus classes - or in other science classes where they are very applicable!

We now set down the general procedure for Euler's method. Again, we must be given  $f'(t)$  and  $f(0)$ . Our goal is to approximate some final value of  $f(t_f)$  by breaking the interval  $(0, t_f)$  into a number of subintervals. We will use the notation  $\Delta t$  to stand for the size of a subinterval. The symbol  $\Delta$  is the Greek capital letter "Delta", and is often used to represent "difference" or change in mathematics.

For example the equation  $f(0 + h) \approx 10 + 2 * h$  we used above might be written using the  $\Delta x$  instead of  $h$  for the small change in  $x$  giving us:  $f(0 + \Delta x) \approx 10 + 2 * \Delta x$ .

Given  $f(0)$  and  $f'(t)$  for all values of  $t$ , here is the method for approximating  $f(t_f)$  using an  $n$  step Euler's method. If we are to have  $n$  steps then each subinterval must be of length  $\Delta t = \frac{t_f - 0}{n}$ . We approximate  $f(t)$  one after the other as follows:

$$\begin{aligned} f(\Delta t) &\approx f(0) + f'(0) * \Delta t \\ f(2\Delta t) &\approx f(\Delta t) + f'(\Delta t) * \Delta t \\ f(3\Delta t) &\approx f(2\Delta t) + f'(2\Delta t) * \Delta t \\ f(4\Delta t) &\approx f(3\Delta t) + f'(3\Delta t) * \Delta t \end{aligned}$$

$$\vdots$$

$$f(t_f) = f(n\Delta t) \approx f((n-1)\Delta t) + f'((n-1)\Delta t) * \Delta t$$

Let's look at an example. Suppose we know that  $f'(t) = 3t^2 - 2$  and that  $f(0) = 15$ . Let's use this to calculate  $f(t)$  between  $t = 0$ , and  $t_f = 4$  by using a 8 step Euler's method. We work by filling in the following table.

$t$	$f(t)$	$f'(t) = 3t^2 - 2$	$f(t + \Delta t)$
0	15	-2	$f(.5) = f(0) + .5f'(0) = 15 + .5 * (-2) = 14$
.5	14		$f(1) = f(.5) + .5f'(.5) =$
1			$f(1.5) = f(1) + .5f'(1) =$
1.5			
2			
2.5			
3			
3.5			
4			

See if you can complete the entries. There is a lot of calculation involved here. In the next section we use the computer to do this work for us.

### 1.3 QuickBasic

QUICKBASIC is one of the programs available under the calculus menu. BASIC is a programming language, and QuickBasic is the version available to us on the Smith machines. To use BASIC you type in a program, which is just a list of instructions telling the machine what you want it to do. Then you "run" the program - which means you tell the computer to go ahead and do what you asked!

Here are some programs to practice with:

Program 1:

```
a= 5+7
PRINT a
```

Program 2:

```
born = 1981
age = 2000-born
PRINT "My age is" age
```

Program 3:

```
FOR i = 1 TO 5
  PRINT i
NEXT i
```

```

Program 4:
FOR i= 1 TO 5
    power= 2^ i
    PRINT power
NEXT i

```

```

Program 5:
sum = 0
FOR i= 1 TO 10
    sum = sum+i
    PRINT "sum of first" i " numbers =" sum ;
NEXT i

```

You will need to know the notation that BASIC uses to denote arithmetic operations. Plus, minus, times, divided by, and to the power are denoted by +, -, \*, /, ^, respectively. There are also some other built in functions, these must be written with capital letters as shown here.

SIN[x]; COS[x]; TAN[x]; SQR[x];LOG[x]

The trig functions are in radians, and the log function is actually the natural log (ln). What do you think SQR[x] does? Try it to see if you are correct.

**Practice Problems:** Try to write programs to do the following: (i) Write your name 35 times. (ii) calculate the square root of each integer from 1 to 100. (iii) Write the integers from 100 down to 1 in decreasing order.

We are now ready to write a program for Euler's Method. The program below, called EULER1.BAS, should be available shortly under the menu for QuickBasic. If it is not listed in the menu of basic programs, you might have to change directories to

O:\COURSES\MTH\CALCULUS\DATA\EULER1.BAS

Or, you can type it in yourself.

```

EULER1.BAS:
tinitial = 0
tfinal = 4
numberofsteps = 8
deltat = (tfinal - tinitial)/numberofsteps
y = 15
FOR k = 1 to numberofsteps
    yprime= 3*t^ 2 - 2
    y = y + yprime * deltat
    t = t + deltat
    PRINT "t=" t "y=" y
NEXT k

```

The program EULER1.BAS is written to calculate  $y(t)$  for  $t$  values up to  $t_f = 4$  starting with the knowledge that  $y' = 3t^2 - 2$  and  $y(0) = 15$  and using 8 steps. What would you need to do to modify this program to solve other problems? (see the exercises)

## 1.4 Graphing $f(t)$ given its derivative.

In the previous sections we were mainly worried about learning one final value of  $f(t_f)$  from our initial conditions. Notice, though that the process of getting  $f(t_f)$  we actually learn lots of intermediate values for  $f(t)$ . In fact we are approximating the function  $f(t)$  between  $t = 0$  and  $t = t_f$  by a series of little straight lines. It would be nice if we could get a visual sense of  $f(t)$  throughout this range rather than just learn the final  $f(t_f)$  value.

We could modify our program EULER1.BAS to do this, by learning how to make QuickBASIC graph for us. But instead we will use an existing program called SLINKY. SLINKY uses a method similar to Euler's method to graph a function from its derivative. To use SLINKY you must only put in your derivative  $f'(t)$  and the initial value of  $f(0)$ . The program does not need to know how many subintervals you want to use or where to stop - it has made the first decision for you, and will just keep going until you tell it to stop later on!

## 1.5 Population Models

When describing population growth, we often use percentages rather than absolute numbers. For example the population of the world is now about 5.3 billion people and growing at a rate of about 2% per year. If  $P(t)$  is the population of the world in year  $t$  then we are saying that  $P'(t) = .02P$ . The derivative in this case is a function of the current population value.

Populations (and other things) often have rates of change that are directly proportional to the current population. That is we have  $P'(t) = kP(t)$  for some constant  $k$ .

**Example:** The population of Mexico in the 1980's grew at a rate proportional to the current population. The population in 1980 was  $P(0) = 67.38$  million and the population in 1981 was  $P(1) = 69.13$  million. Since  $69.13 = (1.026)67.38$ , we see that the growth was *about* 2.6% per year. So we model this as  $P'(t) = .026P(t)$ .

While EULER1.BAS and SLINKY can be used to deal with this situation, we can actually find an exact formula for  $P(t)$  in this case as well. If  $P'(t) = kP(t)$  then  $P(t) = ce^{kt}$ , where the constant  $c$  is determined by the initial conditions.

**Example:** If  $P'(t) = .026P(t)$  and  $P(0) = 67.38$  then  $P(t) = 67.38e^{.026t}$ .

## 1.6 Problems for Chapter 1:

### Exercise 1.1.

Suppose  $f(x) = x^2 - 7x + 4$ . Find the equation of the line tangent to the function  $f$  when  $x = 2$ . Use the graphing program to produce a sketch of  $f$  and its tangent line. If you repeatedly zoom at the point of tangency, what do you observe?

**Exercise 1.2.** Suppose  $f(x) = 2^x$ . Use the graphing utility to obtain a sketch of the function near  $x = 0$ . Repeatedly zoom until  $f$  looks like a straight line through  $(0, 1)$ . Use the mouse to determine an approximation to the slope of this almost straight line. Find the equation for the tangent line of  $f(x)$  at  $x = 0$ .

**Exercise 1.3.** Repeat the procedure for  $f(x) = 3^x$ , and  $f(x) = 4^x$ .

**Exercise 1.4.** Modify EULER1.BAS so that there are 20 steps.

**Exercise 1.5.** Modify EULER1.BAS so it finds  $y(20)$  using 100 steps.

**Exercise 1.6.** EULER1.BAS so that it solves the apple problem from this set of notes. Verify the results we got by hand using 1, 2 and 4 step Euler's methods.

**Exercise 1.7.** Use Euler's method to *approximate* values of  $y(t)$ . Suppose  $y'(t) = 24t^3$  and  $y(0) = 3$ . Estimate  $y(1)$  in each of the following ways.

- (i) Without using the computer, calculate a 1 step approximation.
- (ii -iii) Again without the computer calculate 2 and 4 step approximations.
- (iv - vi) Use the program EULER1 .BAS to find a  $j$ -step approximation to  $y(1)$  for  $j = 10, 100,$  and  $500$ .
- (vii) In fact, you can determine the exact value for  $y(1)$  by computing the antiderivative of  $y'(t) = 24t^3$ , with initial value  $y(0) = 3$ . Do this now. How does it compare with the values you got in the earlier parts of this problem?

**Exercise 1.8.** Poland's population grows at a rate of 9 people per thousand per year while Afghanistan has a growth rate of 21.6 people per thousand per year. In 1985 these countries contained 37.5 million and 15 million people respectively.

- (i) Give formulas for  $P'$  and  $A'$  the derivatives of the populations of Poland and Afghanistan, respectively.
- (ii) Use EULER1 .BAS to predict the population of each of these countries in 2000.
- (iii) Use SLINKY to predict when these two countries will have the same population.
- (iv) Now find formulas for the two populations by taking antiderivatives.
- (v) Use GRAPH, and the formulas from (iv) to predict when the two countries will have the same population.

**Exercise 1.9.** Suppose we assume that a certain bacteria grows at a rate directly proportional to the size of the current population. An experiment begins with 1000 bacteria and one hour later the count is 1500.

- (i) Give a formula for  $y'(t)$  as a function of  $y$ . (Be sure to include units.)
- (ii) Use EULER1 .BAS to predict how many bacteria will be present after one day.

**Exercise 1.10.** A more realistic model of population growth might take into account the limited resources available to a population. In other words, there is most likely some level beyond which the environment cannot sustain the population. One model for this is called a **logistical model**. If  $P(t)$  represents the population at time  $t$ , then we might have

$$P'(t) = .2P(t)\left\{1 - \frac{P(t)}{200,000}\right\}$$

What is the rate of growth when (i)  $P(t) = 10,000$ ; (ii)  $P(t) = 100,000$ ; (iii)  $P(t) = 200,000$ ; (iv)  $P(t) = 300,000$ ?

- (v) Describe in words what the effect of the factor  $\left\{1 - \frac{P(t)}{200,000}\right\}$  is. This is called the logistical factor.
  - (vi) Use SLINKY to sketch the growth of  $P(t)$  over time. Starting with the initial population of 10,000.
  - (vii) Use SLINKY to sketch the growth of  $P(t)$  over time. Starting with the initial population of 500,000.
- Do not print out your graphs, rather, describe what occurs. You may make a rough sketch of the graphs in your homework.

**Exercise 1.11.** A certain bacteria is observed to have a growth rate of  $b'(t) \approx .1b(t)$  for small values of  $b(t)$ . 5,000 of these bacteria are placed in a Petri dish which is assumed to be able to "hold" 500,000.

- Use a logistic factor similar to the previous problem to model this population.
- Use SLINKY to follow the size of the population for awhile (how long is appropriate?).

**Exercise 1.12.** The use of pesticides, particularly DDT has had a drastic effect on the birth rates of several species of birds. One of the best document cases is the plight of the peregrine falcon. The extensive use of DDT began in 1946 and a sharp decrease in the peregrine population was noted the very next year. DDT was used in the US from 1946-1971 and by the end of that period the peregrine was almost extinct. Research has since shown that the DDT caused the birds to lay eggs with a thinner shell, which broke much more easily. Thus, the death rate of the peregrines remained more or less the same as before the use of DDT, but the rate

of *live births* decreased significantly. In 1946 there were 1,000 falcons in the Berkshires. In 1947 there were only 900.

- (i) If  $P$  denotes number of peregrines. Find an appropriate value for  $a$  such that  $P' = aP$ ;
- (ii) Now find a formula for  $P$ .
- (iii) Use your formula to determine how many peregrines should be left after 23 years use of DDT.
- (iv) Use GRAPH to determine how many years it takes for the peregrine population to reach half its original size.

**Exercise 1.13.** The decay of a radioactive material is usually expressed in terms of *half-life*. This is the time required for a quantity of the element to decrease by a factor of one half. If the half life of a material is  $n$  and  $M(t)$  is the amount of the material in year  $t$  then  $M(t) = M(0)e^{-at}$ , where  $n = \frac{\log(2)}{a}$ .

- (i) Verify that  $M(n) = .5M(0)$ .
- (ii) Radiocarbon has a half life of 5600 years. It is what is used in "carbon dating" which has had a significance to archaeologists and other scientists. For example if a piece of coal is found which has only half the radioactivity of a living tree, then we deduce that the tree died about 5600 years ago.  
Give a formula for  $C(t)$ , the amount of radioactivity in carbon.
- (iii) A carved stick was found in the ruins of an ancient civilization. It was tested as having only 40% of the radioactivity of a living tree. How long ago was the civilization?
- (iv) What is the half-life for the peregrine falcons from the previous problem?