

# BOUNDS FOR PARTIAL LIST COLOURINGS

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**Abstract:** Let  $G$  be a simple graph on  $n$  vertices with list chromatic number  $\chi_l = s$ . If each vertex of  $G$  is assigned a list of  $t$  colours Albertson, Grossman and Haas [1] asked how many of the vertices,  $\lambda_{t,s}$ , are necessarily colourable from these lists? They conjectured that  $\lambda_{t,s} \geq tn/s$ . Their work was extended by Chappell [2]. We improve the known lower bounds for  $\lambda_{t,s}$ .

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## §1. Introduction.

Let  $G$  be a simple graph on  $n$  vertices. Suppose that each vertex  $x \in V(G)$  is assigned a list,  $l(x)$ , of possible colours. A proper colouring  $c: V(G) \rightarrow R$  is a *list colouring* of  $G$  if  $c(x) \in l(x)$  for all  $x \in V(G)$ . The graph  $G$  is *s-choosable* if there exists a list colouring for every assignment of lists of  $s = |l(x)|$  colours. The *list chromatic number*, or *choice number*, of  $G$ ,  $\chi_l$ , is the minimum  $s$  for which  $G$  is  $s$ -choosable. Clearly  $\chi_l \geq \chi$ , the chromatic number of  $G$ .

Albertson, Grossman and Haas [1] asked the following: if each vertex of  $G$  is

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assigned a list of  $t \leq s = \chi_l$  colours, how many of the vertices,  $\lambda_{t,s}$ , are necessarily colourable from these lists? They conjectured that  $\lambda_{t,s} \geq tn/s$  and found bounds on  $\lambda_{t,s}$  in different situations. Their work was extended by Chappell, [2]. In this note we establish some further inequalities involving  $\lambda_{t,s}$ .

## §2. Inequalities.

A basic and important result in [1] is the following, a different proof of which we include for the sake of completeness.

**Theorem 1** If  $G$  is a graph with  $n$  vertices and chromatic number  $\chi$  and list chromatic number  $s = \chi_l$ , then

$$\lambda_{t,s} > (1 - (\frac{\chi-1}{\chi})^t)n.$$

**Proof:** Any proper  $\chi$ -colouring of  $G$  partitions the vertices into independent sets  $C_1, C_2, \dots, C_\chi$ . Consider (the set of) partitions of the set of available colours,  $R = \{1, 2, \dots, r\} = \cup l(x)$ , into  $\chi$  classes  $R_1, R_2, \dots, R_\chi$ , where in any such partition a colour goes into a particular class with probability  $1/\chi$ . We now colour a vertex  $x \in C_j$  with a colour from its list of  $t$  colours,  $l(x)$ , only if  $l(x) \cap R_j \neq \emptyset$ ; such a vertex is not coloured with probability  $(1 - 1/\chi)^t$  and the result follows (some partition of  $R$  is at least average).

Theorem 1, as pointed out in [1], is asymptotically correct for any given  $\chi$ .

The chromatic number  $\chi$  naturally plays an important role in the study of  $\lambda_{t,s}$ . For example, suppose that  $s = pt + k$ ,  $0 \leq k < t$ . As some colour class in a proper colouring has at least  $n/\chi$  vertices,  $\chi \leq p$  implies that  $\lambda_{t,s} \geq n/\chi \geq tn/s$  as required. When  $t = 2$ , as pointed out in [1], the lower bound in Theorem 1 exceeds  $2/\chi_l$  whenever  $\chi < \chi_l$ , thus  $\lambda_{2,s} \geq 2n/s$  as conjectured.

In a similar vein to Theorem 1 we have

**Theorem 2** If  $G$  is a graph with  $n$  vertices and list chromatic number  $s = \chi_l = tp$  then  $\lambda_{t,s} \geq tn/s$ .

**Proof** For each vertex  $x \in V(G)$ , we augment its list,  $l(x) = \{x_1, x_2, \dots, x_t\}$ , to create a list of  $pt$  colours by adding  $p-1$  copies of  $l(x)$ :  $l^2(x) = \{x_1^1, x_2^1, \dots, x_t^1, \dots, x_1^p, x_2^p, \dots, x_t^p\}$ . Since  $G$  is  $s$ -choosable we may find a proper colouring,  $\phi$ , using the augmented lists. This colouring  $\phi$  defines a partition of  $V(G)$  into list-coloured classes  $L_1, L_2, \dots, L_p$  according to the use of colours from the list

$l(x)$  or its  $p-1$  copies. One of these classes contains at least the average number of vertices, thus the  $t$ -lists on the vertices of  $G$  are sufficient to list colour at least  $n/p = tn/s$  vertices.

**Corollary** If  $G$  is a graph with  $n$  vertices and list chromatic number  $s = \chi_l$

where  $s = tp + k$ ,  $0 < k < t$ , then  $\lambda_{t,s} \geq \frac{n}{\lceil \frac{s}{t} \rceil}$ .

**Proof** Note that  $\lceil s/t \rceil = p+1$ . Using the same proof as in the above theorem except using copies of  $l(x)$  to augment the lists to size  $(p+1)t$ , we are done since  $s$ -choosability implies  $(s+t-k)$ -choosability.

Notes:

1. It follows from the above arguments that  $p\lambda_{t,s} \geq \lambda_{pt,s}$ .
2. It is tempting to think that the above proof could be extended by adding only a list of size  $k$  as the last list that covers  $k/t$  times as many vertices as the lists of size  $t$ . However it is not too difficult to construct an example in which lists of size  $m$  do not cover more than lists of size  $m-1$ , i.e. we cannot assume that the function is strictly increasing.
3. Further to point 2, it is possible that the colours from the list of size  $k$  can be assigned to more than  $k/t$  times as many vertices as those from the lists of size  $t$ . Thus, since there are not the same number of copies of each colour, the averaging argument from the last sentence in the proof of Theorem 2 may fail.

In [2], Chappell showed in general that  $\lambda_{t,s} > (6/7)tn/s$ . By the Corollary we have, when  $s = tp + k$ ,  $0 < k < t$ , that

$$\lambda_{t,s} \geq \frac{p+k}{p+1} \frac{tn}{s},$$

which is an improvement for  $s > 6(t-k)$  (or simply, for  $p \geq 6$ ). Chappell's main result is that  $\lambda_{t,s} \geq qn$ , where  $q$  is the unique positive root of the equation  $1 - x - (1 - (1-x)/(s-t))^t = 0$ . This may be improved slightly in certain cases by combining his proof with that of the above.

**Theorem 3** If  $G$  is a graph with  $n$  vertices and list chromatic number  $s = \chi_l$  where  $s = tp + k$ ,  $0 < k < t$ , then  $\lambda_{t,s} \geq qn$  where  $q$  is the unique positive root of  $1 - x - (1-(1-px)/k)^t = 0$ .

**Proof** For each vertex  $x \in V(G)$  augment its list  $l(x)$ ,  $\{x_1, x_2, \dots, x_t\}$  to create a list of size  $s$  by adding  $p-1$  copies of  $l(x)$ :  $l^2(x), \dots, l^p(x)$ , as before, and also by adding the following list of  $k$  new colours:  $\{z_1, z_2, \dots, z_k\}$ . Since  $G$  is now choosable we may find a proper colouring,  $\phi$ , using the augmented lists. This colouring  $\phi$  defines a partition of  $V(G)$  into list-coloured classes  $L_1, L_2, \dots, L_{p+k}$  according to the use of colours from the list  $l(x)$  or its  $p-1$  copies, and  $k$  classes of independent vertices corresponding to the use of colours from the final set of  $k$  colours. Partition the original set of colours,  $R = \{1, 2, \dots, r\}$ , randomly into  $p+k$  classes  $R_1, R_2, \dots, R_{p+k}$  where, for  $1 \leq j \leq p$ ,  $\text{prob}(\text{colour } i \in R_j) = q$ , and, for  $1 \leq j \leq k$ ,  $\text{prob}(\text{colour } i \in R_{p+j}) = (1-pq)/k$ . (Re)colour  $G$  as follows: a vertex in  $L_j$ ,  $1 \leq j \leq p$  is coloured by colour  $x_i$ , if it was coloured by  $x_i$  or by a copy of  $x_i$  under  $\phi$ , and if also  $x_i \in R_j$ ; a vertex in  $L_{p+j}$ ,  $1 \leq j \leq k$ , is coloured with a colour from its list if its  $t$ -list intersects  $R_{p+j}$ . Otherwise a vertex is not coloured. In this scheme, we have  $\text{prob}(x \in L_j, 1 \leq j \leq p, \text{ is coloured}) = q$  and  $\text{prob}(x \in L_{p+j} \text{ is not coloured}, 1 \leq j \leq k) = 1 - ((1-pq)/k)^t$ . Then all vertices of  $V(G)$  are coloured with probability  $q$  if  $f(q) = 1 - (1-(1-pq)/k)^t - q = 0$  and the result follows.

Note: the above  $f(q)$  is a decreasing function and, for  $s = pt+k$ ,  $f(t/s) =$

$$1 - \frac{t}{s} - \left(1 - \frac{1}{s}\right)^t = -\binom{t}{2}/s^2 + \binom{t}{3}/s^3 - \binom{t}{4}/s^4 + \dots + (-1)^{t+1} \binom{t}{t}/s^t < 0.$$

The advantage over Chappell's result appears when  $p > 1$ . A typical implication of Theorem 3 is

**Corollary** For  $s = 2p+1$ ,  $\lambda_{2,s} > \frac{-1 + \sqrt{4p^2 + 1}}{2p^2} n$ .

Thus for example  $\lambda_{2,5}$  is bounded below by  $(-1+17^{1/2})n/8$  instead of  $(-13+189^{1/2})n/2$  from Chappell's result (about  $.3904n$  instead of  $.3739n$ ).

Note: We have learned (private communication) that Theorem 2 was also obtained by Chappel.

## References

1. M. O. Albertson, S. Grossman, and R. Haas, Partial List Colorings, *Discrete Math.*, 214 (2000) 235-240.
2. G. C. Chappell, A Lower Bound for Partial List Colorings, *J. Graph Theory*, Vol. 32, 4, 1999, 390-393.