Star Forests, Dominating Sets and Ramsey-type Problems

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Abstract

A *star forest* of a graph G is a spanning subgraph of G in which each component is a star. The minimum number of edges required to guarantee that an arbitrary graph, or a bipartite graph, has a star forest of size n is determined. Sharp lower bounds on the size of a largest star forest are also determined. For bipartite graphs, these are used to obtain an upper bound on the domination number in terms of the number of vertices and edges in the graph, which is an improvement on a bound of Vizing. In turn, the results on bipartite graphs are used to determine the minimum number of lattice points required so that there exists a subset of n lattice points, no three of which form a right triangle with legs parallel to the coordinate axes.

Key words: star forests, dominating sets, Ramsey theory,

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1 Introduction

A graph is a *star* if it is isomorphic to $K_{1,n}$ for some non-negative integer n. The *center* of a star S with $|V(S)| \neq 2$ is its unique vertex of maximum degree. Either vertex of $K_{1,1}$ (but not both vertices simultaneously) can be its *center*. If a vertex x is the center of a star S, we say S is *centered* at x. A *star forest* of a graph G is a

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spanning subgraph of G in which each component is a star. The *size* of a star forest is the size of its edge set. As in any forest, this equals the number of vertices of G minus the number of components in the star forest. The largest size of a star forest of G is denoted by $\sigma(G)$, which is abbreviated to σ when the graph G is clear from the context.

Star forests have previously appeared in the literature in work on star arboricity. The *star arboricity*, $st(G)$, of a graph is the minimum number of star forests whose union covers all edges of G. It was perhaps first studied by Akiyama and Kano (3). There are bounds for $st(G)$ for several classes of graphs including regular graphs and planar graphs, see for example (4) and (8). Clearly, for a graph G, $\sigma(G) \geq \frac{|E|}{st(G)}$.

Each star forest of G corresponds to a dominating set of G, since a set of centers of the stars in a star forest is a dominating set of G. On the other hand, if X is a dominating set of G, then every vertex in $V - X$ is adjacent to a vertex in X. The graph G therefore has a spanning subgraph each component of which is a star centered at a vertex in X. That is, every dominating set corresponds to at least one star forest. Therefore, $\sigma(G) = |V| - \gamma(G)$, where $\gamma(G)$ denotes the domination number of G. Complexity results for the decision problem “Does the given graph G have a star forest of size at least the given integer k?” follow immediately from this observation and complexity results for the domination number. In particular, the problem of determining σ is NP-complete for bipartite graphs. The corresponding result for the domination number has been proved independently by several authors (see (7), page 301).

A motivating Ramsey-type question from combinatorial geometry will be discussed in Section 2. Its solution, based on our graph theoretic results, is presented in the final section. In Section 3, Vizing’s bound on the domination number of a graph ((10), also see (7), page 55) is used to determine the number of edges required to guarantee the existence of a star forest of size n. Star forests of bipartite graphs are studied in Section 4. A sharp bound on σ is determined first, and then used to solve the analogous problem restricted to bipartite graphs. This result leads to an improvement on Vizing’s bound for the domination number.

2 Point sets in the plane

Many problems in combinatorial geometry concern properties of point sets in the plane. Examples include determining the number of times a given distance can occur, or how many unit area triangles can occur, in an k point set in the plane. See (2) for a good overview.

In this paper we consider the following Ramsey-type problem: *Determine $r(n)$, the smallest integer such that given any collection of $r(n)$ points, there exist n points no* *three of which are the vertices of a right triangle.* An upper bound, $r(n) \leq 1 + (n-1)^2$, follows from the classic result of Erdös and Szekeres (5) that any sequence of $n^2 + 1$ distinct integers contains an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$. Suppose $n^2 + 1$ points in the plane are given. Introduce axes so that the x-coordinates are all distinct. The sequence of y-coordinates thus has a monotone subsequence of length $n + 1$, and no three of the corresponding points can be the vertices of a right triangle. For $n = 4$ there is a lower bound as well (personal communication from A.C. Liu) namely, $8 \leq r(4)$. This is proved by considering a 3×3 array of points with the top right and left corner points removed.

Our results in Section 5 will provide a quadratic lower bound for $r(n)$ for all n, thus determining its order of magnitude. We do this by restricting our point set to the lattice, and considering only triangles with legs parallel to the coordinate axes. This same approach was used in (9) to prove the result mentioned below. Explicitly, we define a T-set to be a collection of three lattice points in the plane that form a right triangle with legs parallel to the coordinate axes. Let $T(n)$ be the smallest integer such that, given any collection of $T(n)$ lattice points, there exist n not containing a T-set. The problem of determining $T(n)$ is solved in Section 5 using our results about star forests in bipartite graphs.

Pach and Sharir (9) proved that the maximum number of times any angle $0 < \alpha < \pi$ can occur between the ordered triples of k points in the plane is $O(k^2 log(k))$. Furthermore, for infinitely many values of alpha there exists a constant $c = c(\alpha)$ and an k- element point set so that at least $ck^2 \log(k)$ triples determine the angle α. Their construction does not work for finding a k point set with $ck^2 \log(k)$ triples determining the special angle $\alpha = \pi/2$.

Another related problem was considered by Abbott and Hanson (1). Define an *S-set* to be a set of four lattice points which are the vertices of a square with sides parallel to the axes. Define $q(n)$ to be the smallest integer such that, given an $n \times n$ square array of lattice points, there exist $q(n)$ points which do not contain an S-set, but which are such that the addition of any new point causes the appearance of an S-set. Abbott and Hanson proved that for $n \geq 2$, $q(n) < n^\alpha$ for some $\alpha < 2$.

3 Star Forests in Graphs

Let $s(n)$ be the smallest integer such that any graph with $s(n)$ edges has a star forest of size at least n. In this short section we determine $s(n)$ for any positive integer n. This follows easily from a theorem of Vizing.

Theorem 1 (Vizing (10), also see (7), page 55) *For any graph G,*

$$\gamma \leq |V| + 1 - \sqrt{1 + 2|E|}.$$

Corollary 2 *For any* $n \geq 1$, $s(n) = \lfloor \frac{n^2+1}{2} \rfloor$.

PROOF. We show first that $s(n) \geq \lfloor \frac{n^2+1}{2} \rfloor$ by exhibiting graphs with $\lfloor \frac{n^2-1}{2} \rfloor = \lfloor \frac{n^2+1}{2} \rfloor - 1$ edges and $\sigma < n$. If n is odd, then let $G = K_{n+1} - F$, where F is a 1-factor, and if n is even then let G be the graph constructed from $K_n - F$ by adding a new vertex and joining it to $n - 1$ of the original vertices.

To complete the proof, let G be a graph with $|E| \geq \lfloor \frac{n^2+1}{2} \rfloor$. We show that $\sigma(G) \geq n$. By Theorem 1, $\gamma \leq |V| + 1 - \sqrt{1 + 2\lfloor \frac{n^2+1}{2} \rfloor}$. Hence, $\sigma = |V| - \gamma \geq \sqrt{1 + 2\lfloor \frac{n^2+1}{2} \rfloor} - 1$. If n is odd then the right hand side is $\sqrt{n^2 + 2} - 1$, and if n is even then the right hand side is $\sqrt{n^2 + 1} - 1$. Since σ is an integer, the result follows. ∎

In (6) Fulman gives an improvement of Vizing's bound by considering $|V|$, $|E|$ and Δ. This bound can also be used to determine a bound on $\sigma(G)$.

4 Star Forests in Bipartite Graphs

In this section we solve, for bipartite graphs, the analogous Ramsey-type problem as in the previous section. Whereas, for arbitrary graphs, the proof used Vizing's Theorem, here we establish the Ramsey-type theorem first and then use it to obtain an improvement of Vizing's Theorem for bipartite graphs. We then investigate whether further information about the given bipartite graph is helpful in obtaining lower bounds on the size of a star forest.

Lemma 3 *If B is a bipartite graph, then*

$$\sigma(B) \geq \Delta - 2 + \left\lceil \frac{|E|}{\Delta} \right\rceil.$$

PROOF. Suppose B has bipartition (S, T), where S contains a vertex v of degree Δ. Let $X = T - N(v)$, and $Y = S - (\{v\} \cup N(X))$. The number of edges incident with vertices in $\{v\} \cup N(X)$ is at most $\Delta(1 + |N(X)|)$. The remaining edges join vertices in Y to vertices in $N(v)$. Therefore, some vertex w in $N(v)$ is joined to at least

$$\left\lceil \frac{|E| - \Delta(1 + |N(X)|)}{\Delta} \right\rceil$$

vertices in Y. Construct a star forest $\mathcal{S}$ consisting of the edges incident with v except vw, all edges joining w to vertices in Y, and one edge joining each vertex in $N(X)$ to a vertex in X. The number of edges in $\mathcal{S}$ is at least

$$(\Delta - 1) + \left\lceil \frac{|E| - \Delta(1 + |N(X)|)}{\Delta} \right\rceil + |N(X)| = \Delta - 2 + \left\lceil \frac{|E|}{\Delta} \right\rceil. \quad \blacksquare$$

Corollary 4 *For any bipartite graph G,*

$$\gamma \leq |V| - \Delta - \left\lceil \frac{|E|}{\Delta} \right\rceil + 2. \quad \blacksquare$$

Examples of graphs for which the above bounds are sharp include $K_{n,n}$ and the graphs described in the proof of Theorem 9.

We now turn our attention to the Ramsey-type problem. Define $s_B(n)$ to be the smallest integer such that any bipartite graph with at least $s_B(n)$ edges has a star forest of size at least n.

Theorem 5 *For any $n \geq 3$, $s_B(n) = 1 + \lfloor \frac{(n+1)^2}{4} \rfloor = 1 + \lfloor \frac{(n+1)}{2} \rfloor \lceil \frac{(n+1)}{2} \rceil$.*

PROOF. Let B be a bipartite graph with $|E| \geq 1 + \lfloor \frac{(n+1)^2}{4} \rfloor$. By Lemma 3, $\sigma(B) \geq \Delta - 2 + \left\lceil \frac{\lfloor \frac{(n+1)^2}{4} \rfloor + 1}{\Delta} \right\rceil$. Now,

$$\Delta - 2 + \left\lceil \frac{\lfloor \frac{(n+1)^2}{4} \rfloor + 1}{\Delta} \right\rceil \quad \geq \quad n$$

if and only if

$$\left\lceil \frac{\lfloor \frac{(n+1)^2}{4} \rfloor + 1}{\Delta} \right\rceil + \Delta - 2 - n \quad \geq \quad 0.$$

Using the fact that for all integers a, b and c, $\lceil \frac{a}{b} \rceil \geq c$ if and only if $a + b - 1 \geq bc$, the above inequality becomes

$$\left\lfloor \frac{(n+1)^2}{4} \right\rfloor + 1 + (\Delta - 1) + \Delta(\Delta - 2 - n) \quad \geq \quad 0,$$

which simplifies to

$$\left\lfloor \frac{(n+1)^2}{4} \right\rfloor + \Delta^2 - (n+1)\Delta \geq 0.$$

Since for any integer n, $\lceil \frac{n+1}{2} \rceil \lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{(n+1)^2}{4} \rfloor$, the above inequality becomes

$$\left(\Delta - \left\lceil \frac{n+1}{2} \right\rceil\right)\left(\Delta - \left\lfloor \frac{n+1}{2} \right\rfloor\right) \geq 0,$$

which is true for all integers Δ. Therefore, $s_B(n) \le \lceil \frac{n+1}{2} \rceil \lfloor \frac{n+1}{2} \rfloor + 1$.

To see that equality holds, consider $K_{\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n+1}{2} \rfloor}$. This graph has $\lfloor \frac{(n+1)^2}{4} \rfloor$ edges, and a star forest has at least two components and, therefore, at most $\lceil \frac{n+1}{2} \rceil + \lfloor \frac{n+1}{2} \rfloor - 2 = n - 1$ edges. ∎

We now use Theorem 5 to establish an improvement of Vizing's Theorem (Theorem 1) for bipartite graphs. Let B be a bipartite graph, and n the largest integer such that $1 + \lfloor \frac{(n+1)^2}{4} \rfloor = s_B(n) \le |E|$. Then $s_B(n+1) > |E|$ or, equivalently, $n > -2 + 2\sqrt{|E| - 1}$.

Corollary 6 *For any bipartite graph B with at least one edge,*

$$\gamma < |V| + 2 - 2\sqrt{|E| - 1}$$

and, equivalently,

$$\sigma > 2\sqrt{|E| - 1} \ - 2.$$

PROOF. By Lemma 3 it is enough to show

$$\Delta - 2 + \frac{|E|}{\Delta} > -2 + 2\sqrt{|E| - 1}.$$

This inequality is equivalent to

$$\Delta^4 + 4\Delta^2 - 2\Delta^2|E| + |E|^2 > 0.$$

This quadratic in $|E|$ achieves its minimum at $|E| = \Delta^2$. The minimum value

$$\Delta^4 + 4\Delta^2 - 2\Delta^4 + \Delta^4 > 0$$

for all values of $\Delta > 0$. This completes the proof. ∎

This result extends Vizing's bound for all graphs by considering how far the graph is from being bipartite. Let G be a graph and B be a bipartite subgraph of G with a maximum number of edges, then the bipartite density of G is $b(G) = |E(B)|/|E(G)|$. It is easy to see that $1/2 \le b(G) \le 1$ and $\gamma(G) \le \gamma(B)$. The following corollary affords significant improvement over Theorem 1 when G has small chromatic number.

Corollary 7 *For any graph G, $\gamma < |V| + 2 - 2\sqrt{b(G)|E| - 1}$.*

By Theorem 5, in order to guarantee the existence of a star forest with n edges, a bipartite graph must have more edges than any complete bipartite graph $K_{s,t}$, where $s+t=n+1$. One might ask whether bipartite graphs in which the sets in the bipartition have sizes at least s and t, and the number of edges is greater than $|E(K_{s,t})|$, have $\sigma \geq s+t-1=\sigma(K_{s,t})+1$. If true, this would be a generalization of Theorem 5. By Corollary 6, a bipartite graph with $st+1$ edges has $\sigma > 2\sqrt{st}-2$. The examples in Theorem 9 indicate that, in general, the above conditions are not sufficient to guarantee the existence of a star forest of size $s+t-k$, for any positive integer k. When $k \geq 2$, Corollary 8 gives the best possible range of values for t for which $\sigma \geq s+t-k$.

Corollary 8 *Let $k \geq 2$ be an integer. If B is a bipartite graph with $st+1$ edges and $s \leq t \leq s+2\sqrt{s(k-1)}+k-1$ then $\sigma(B) \geq s+t-k$.*

PROOF. Following the same argument as in the proof of Theorem 5,

$$\Delta - 2 + \left\lceil \frac{st+1}{\Delta} \right\rceil \;\geq\; s+t-k$$

if and only if $\Delta^2-(s+t+1-k)\Delta+st \geq 0$. If t is in the given range, then the discriminant of this quadratic in Δ is non-positive, so the inequality holds. ∎

Observe that the above corollary, in fact, makes no assumption about the size of the sets in the bipartition. However, in most of the examples given below the sets in the bipartition have size $|S| \geq s$ and $|T| \geq t$.

Theorem 9 *For any positive integer $k \geq 2$ there exist infinitely many positive integers s and t such that there is a bipartite graph with at least $st+1$ edges and $\sigma = s+t-k-1$.*

PROOF. Let $s \geq 4k$ be such that $\sqrt{s(k-1)}$ is an integer. Let $t=s+2\sqrt{s(k-1)}+k$, and $z=s+\sqrt{s(k-1)}$. We note first that $K_{z,z+1}$ is an example of such a graph. Let B be any bipartite graph constructed as follows. The bipartition is (S,T), where $|S|=z \geq s$ and $|T|=t$. There is a subset $Z \subseteq T$ such that the subgraph of B induced by $S \cup Z$ is $K_{z,z}$ and each vertex in S is adjacent to exactly one vertex in $T-Z$, with each vertex in $T-Z$ adjacent to at least two vertices in S. The latter condition can be satisfied because $s \geq 4k$.

The number of edges in the graph B is $z^2+z=st+\sqrt{s(k-1)} \geq st+1$. In any star forest of B, each vertex of $T-Z$ must belong to a different component. If each such component is centered in $T-Z$ then, since the vertices in Z must also belong to the star forest, there is at least one component not centered in $T-Z$, for a total of at least $\sqrt{s(k-1)}+k+1$ components. On the other hand, suppose the vertex

$x \in T - Z$ belongs to a component centered at $y \in S$. Let $w \neq y$ be another vertex of S which is adjacent to x. The component of the star forest containing w does not contain any vertex of $T - Z$, since the only possibility is x. Thus, in this case, there are also at least $\sqrt{s(k-1)} + k + 1$ components. The number of edges in the star forest equals the number of vertices of B minus the number of components in the forest, which is at most $s + t - k - 1$. ∎

We conclude this section by addressing the case $k = 1$.

Corollary 10 *Let B be a bipartite graph with $st + 1$ edges. If Δ does not belong to the open interval (s, t), then B has a star forest of size $s + t - 1$.*

PROOF. Following the same argument as in the proof of Theorem 5,

$$\Delta - 2 + \left\lceil \frac{st+1}{\Delta} \right\rceil \geq s + t - 1$$

if and only if $(\Delta - s)(\Delta - t) \geq 0$. ∎

Corollary 10 implies that if $|s - t| \leq 1$ then $st + 1$ edges suffice to guarantee the existence of a star forest of size $s + t - 1$. On the other hand, if s is any positive integer and $t = s + 2$, the complete bipartite graph $K_{s+1,s+1}$ has $st + 1$ edges and $\sigma = s + t - 2$. We note, however, that the bipartition contains no set of size at least t. We do not know the largest value of $t \geq s$ for which any bipartite graph with $st + 1$ edges and sets in the bipartition of size s and t has $\sigma \geq s + t - 1$.

5 Subsets of Lattice Points Containing no Right Triangle

Define a *T-set* to be a set of three lattice points in the plane which are the vertices of a right-angled triangle with legs parallel to coordinate axes. Let $T(n)$ be the smallest integer such that, given any collection of $T(n)$ lattice points, there exist n points not containing a T-set. In this section we apply the results from Section 4 to determine $T(n)$.

Theorem 11 *For all integers $n \geq 3$, $T(n) = \lfloor \frac{(n+1)^2}{4} \rfloor + 1 = s_B(n)$.*

PROOF. Given a collection of lattice points, L, construct a bipartite graph $B = (R, C)$ with a vertex in R for each occupied row, and in C for each occupied column. An edge rc is in $E(B)$ exactly when the lattice point (r, c) is in L. A T-set in

L corresponds to a path of length three in B. Hence, there is a collection of n lattice points in L not containing a T-set if and only if $\sigma(B) \geq n$. The result now follows from Theorem 5. ∎

We note that Corollary 6 asserts that given any collection of $st + 1$ lattice points in the plane there is a subcollection of at least $2\sqrt{st} - 2$ lattice points containing no T-set. Theorem 9 asserts that, in general, one can do no better.

These results about $T(n)$ establish a lower bound and thus the order of magnitude for the function in the original Ramsey-type question.

Corollary 12 *Let $r(n)$ be the smallest integer such that given any collection of $r(n)$ points, there exist n points no three of which are the vertices of a right triangle. For $n \geq 3$,*

$$\lfloor \frac{(n+1)^2}{4} \rfloor + 1 \leq r(n)$$

and $r(n) = \Theta(n^2)$.

PROOF. The inequality follows directly from the fact that $T(n) \leq r(n)$. The order of magnitude of $r(n)$ is now determined from this and the upper bound discussed in the introduction. ∎

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