Mathematics of the Falling Cat

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February 2, 2012

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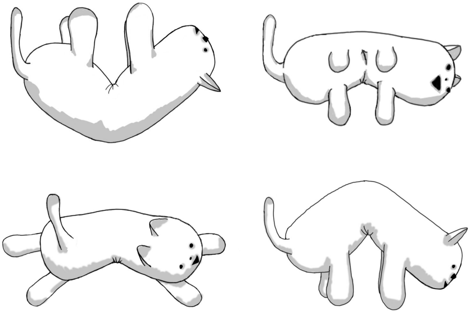
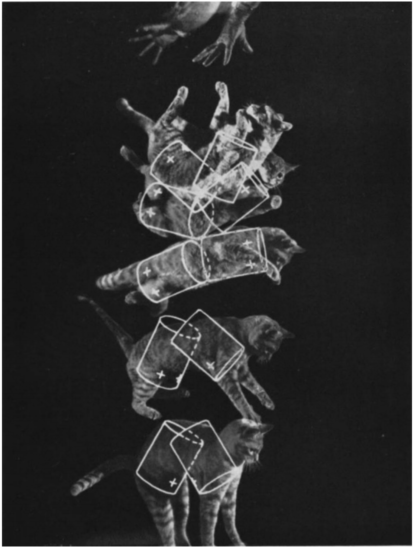
- Rademaker, Ter Braak (1935) - first solution
- Kane, Scher (1969) - more realistic class of solutions
- Montgomery (1993) - full mathematical theory

The mathematical cat

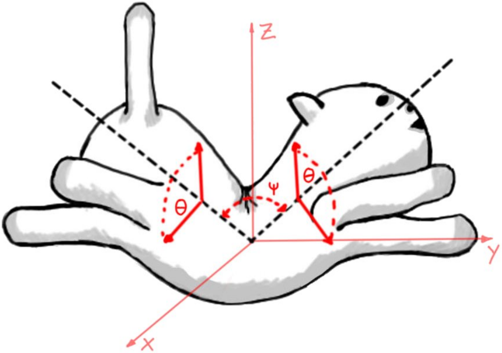
A cat's body is modeled as a pair of equal cylinders, connected by a joint (its spine). The spine can bend, but it does not twist.





The cat's shape

The shape of the cat is given by two angles (ψ, θ).



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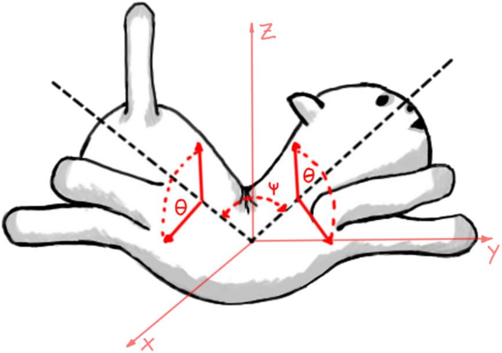

The cat's shape

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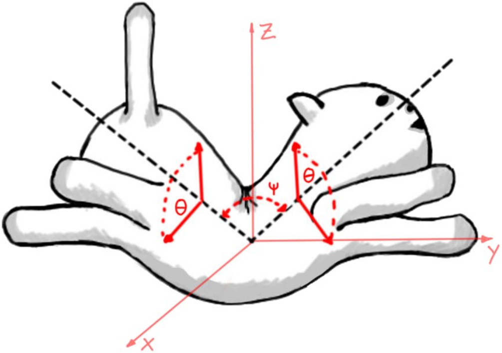
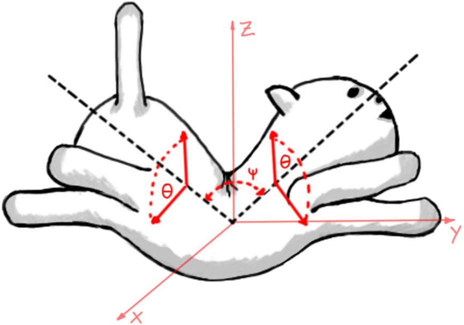
- ψ is the angle between the two halves of the cat's body.
- θ describes the direction of the cat's legs ($\theta = 0$ when the front and back legs are closest to each other). A change in θ corresponds to a rotation of the cat's body around the "spinal axis".





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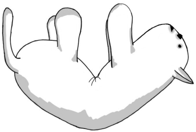

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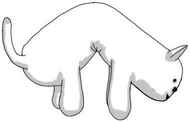

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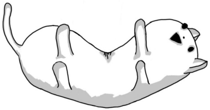

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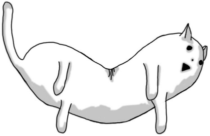

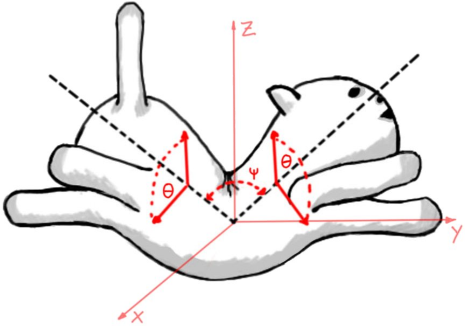

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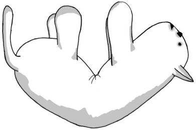
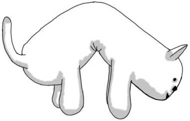
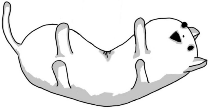
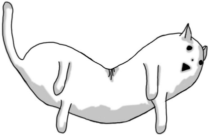
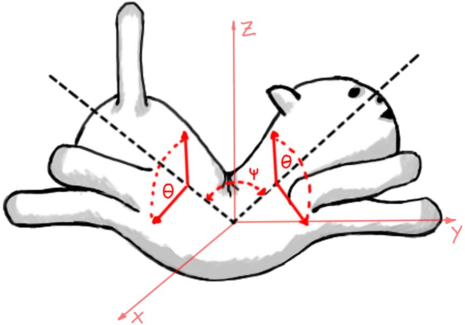
1 

2 

3 

4 



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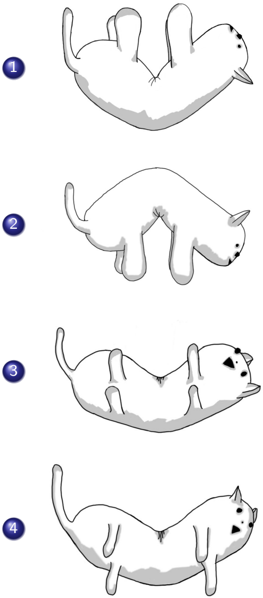

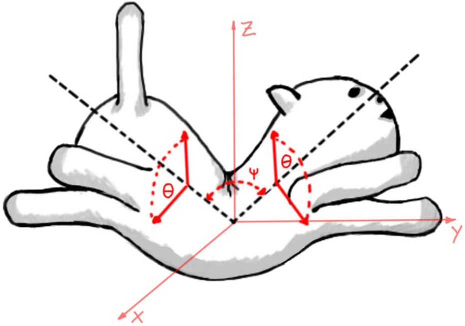

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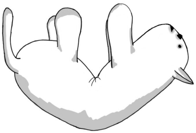
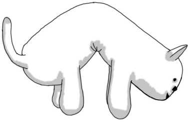
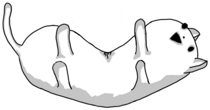
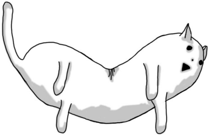
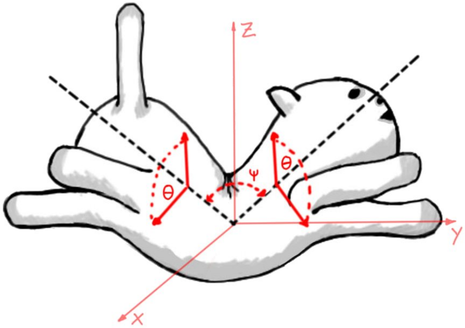
1. 
2. 
3. 
4. 



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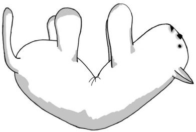
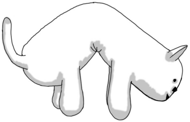
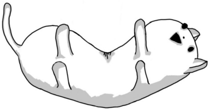
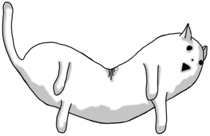
1. 
2. 
3. 
4. 

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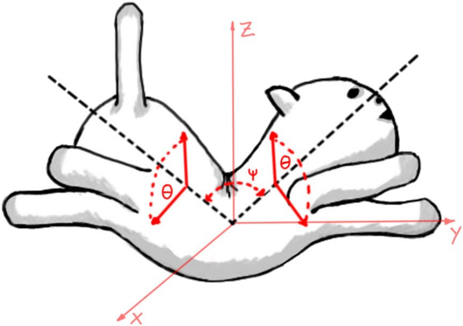
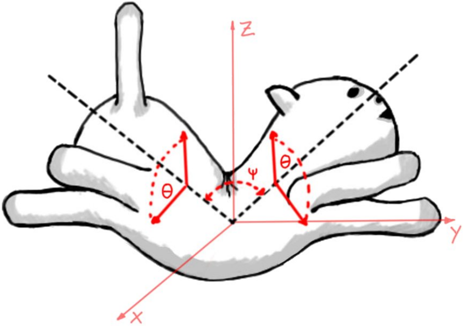
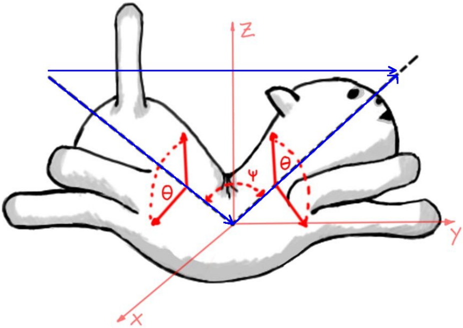
- No angular momentum: If the cat doesn't change its shape, then it will not rotate.
- If the cat changes its shape, then the entire body will rotate to "cancel out" the angular momentum of the shape change.
- We can consider changes in ψ and θ separately.



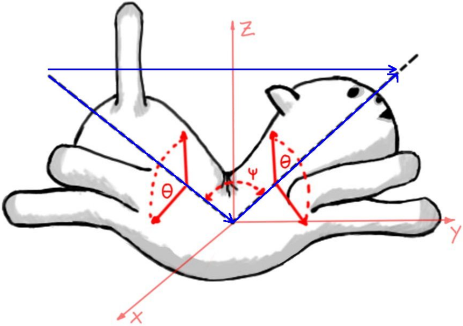
- A change in ψ is "balanced": the front and back halves of the body have opposite angular momentum.



- A change in ψ is "balanced": the front and back halves of the body have opposite angular momentum.
- The cat can change ψ without causing the body to rotate.



- As θ changes, the front and back halves of the body are both rotating about the bent spine.



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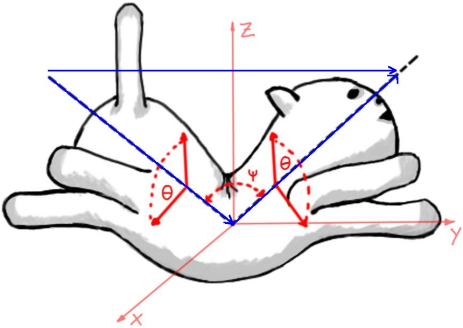
- The total angular momentum vector is parallel to the y-axis.



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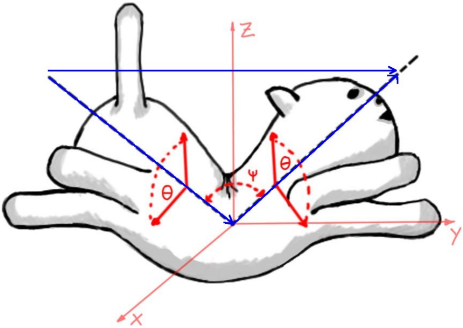

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- The total angular momentum vector is parallel to the y-axis.
- The *size* of the total angular momentum depends on ψ.
- The rate of rotation needed to compensate is

$$\frac{\alpha \sin(\psi/2)}{\cos^2(\psi/2) + \alpha \sin^2(\psi/2)}$$

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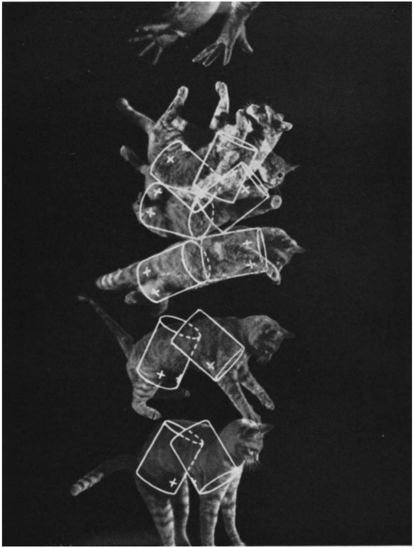
2. It swings its legs around until they are positioned correctly (note that its back is arched at this point).
3. It is now free to curve its back and prepare for landing.

The Kane-Scher solution



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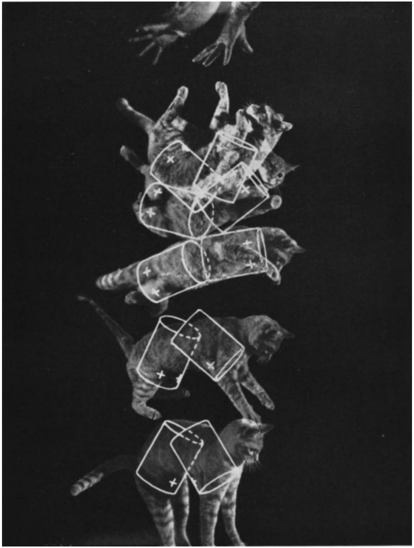

Question

Can you think of a way to drop a cat so it can't land on its feet?

Thanks. (And thanks to Eric Kuehne for the cat drawings)