Homotopy Poisson actions

Rajan Mehta

November 8, 2010
Conventional perspectives

Definition
A *Poisson structure* on a manifold $M$ is a Lie bracket on $C^\infty(M)$ that satisfies the Leibniz rule.

Equivalently,

Definition
A *Poisson structure* on a manifold $M$ is a bivector field $\pi \in \mathfrak{X}^2(M) = \Gamma(\wedge^2 TM)$ such that $[\pi, \pi]_{\text{Schouten}} = 0$.

Derived bracket formula:

$$\{ f, g \}_\pi = [[\pi, f], g].$$
Differential perspective

\( d_\pi := [\pi, \cdot] \) is a degree 1 operator on \( \mathfrak{X}^\bullet(M) = \Gamma(\wedge TM) \).

- \([\pi, \pi] = 0 \iff d_\pi^2 = 0 \) (\( \iff \) Lie algebroid \( T^*M \)).
- \( d_\pi \) is a graded derivation with respect to the wedge product and the Schouten bracket.

Derived bracket formula:

\[ \{f, g\}_\pi = [[\pi, f], g] = [d_\pi f, g] . \]
Graded geometry perspective

\[ \mathfrak{x}^\bullet(M) = \text{algebra of “smooth functions” on } T^*[1]M. \]

\( d_\pi \) is a derivation of the product structure \( \iff \) \( d_\pi \) is a vector field on \( T^*[1]M. \)

- \( d_\pi \) is deg. 1 and \( d_\pi^2 = 0 \iff \) \( d_\pi \) is homological
  \((T^*[1]M, d_\pi) \) is an \( NQ \)-manifold.
- \( d_\pi \) is a derivation of Schouten \( \iff \) \( d_\pi \) is symplectic.

Definition
A Poisson structure on \( M \) is a homological symplectic vector field on \( T^*[1]M. \) \((T^*[1]M, \omega, d_\pi) \) is a deg. 1 symplectic \( NQ \)-manifold.

Definition
A Poisson structure on \( M \) is a degree 2 function \( \pi \) on \( T^*[1]M \) such that \([\pi, \pi] = 0.\)
Cattaneo-Zambon: Poisson reduction = (super)symplectic reduction of $T^*[1]M$

For moment map reduction, they considered DGLA actions. If the comoment map $\mathfrak{g} \to C^\infty(T^*[1]M)$ is a DGLA map, then $\pi$ passes to the quotient.

We also want to include Poisson-Lie group/Lie bialgebra actions.

- dg-group = $Q$-group = (graded) Lie group with multiplicative vector field, $[Q, Q] = 2Q^2 = 0$.
- Poisson-Lie group = Lie group with multiplicative bivector field, $[\pi, \pi] = 0$.
- homotopy Poisson-Lie group = Lie group with multiplicative multivector field, $[\pi, \pi] = 0$. 
Homotopy Poisson manifolds

Let $\mathcal{M}$ be a graded manifold.

**Definition**

A *homotopy Poisson* (*hPoisson*) structure on $\mathcal{M}$ is any of the following equivalent things:

- an $L_\infty$ algebra structure on $C^\infty(\mathcal{M})$ where the brackets satisfy the Leibniz rule.
- a homological symplectic vector field on $T^*[1]\mathcal{M}$.
- a degree 2 function $\pi$ on $T^*[1]\mathcal{M}$ such that $[\pi, \pi] = 0$.

Write $\pi = \sum \pi_k$, where $\pi_k \in \mathfrak{X}^k(\mathcal{M})$. Then we have the derived bracket formula

$$\{f_1, \ldots, f_k\}_\pi = \cdots [[\pi_k, f_1], f_2], \ldots f_k] = \cdots [d_\pi f_1, f_2], \ldots f_k].$$

**Note:** the “homological” degree of $\pi_k$ is $2 - k$. 
Examples

Example
A graded (deg. 0) Poisson manifold is an hPoisson manifold. Note: For ordinary manifolds, then hPoisson = Poisson.

Example
$Q$-manifolds/dg-manifolds, e.g. $A[1]$ if $A$ is a Lie algebroid.

Example
A $QP$-manifold is a Poisson manifold equipped with a homological Poisson vector field, e.g. $T^*(A[1])$ if $A$ is a Lie algebroid.
Another example

Example
If \( \mathcal{V} = \bigoplus V_i[i] \) is an \( L_\infty \)-algebra, then \( \mathcal{V}^* = \bigoplus V_i^*[-i] \) is a (linear) hPoisson manifold. \( T^*[1](\mathcal{V}[1]) = T^*[1](\mathcal{V}^*) \).

Remark
If \( \mathcal{M} \) is hPoisson, then \( T^*[1]\mathcal{M} \) is a degree 1 symplectic Q-manifold, but generally has negative degree coordinates even if \( \mathcal{M} \) is \( \mathbb{N} \)-graded.

c.f. Roytenberg-Severa correspondence

\[
\{ \text{Poisson manifolds} \} \leftrightarrow \{ \text{deg. 1 symplectic NQ-manifolds} \}
\]
Morphisms

Definition
A (strict) morphism of hPoisson manifolds from \((\mathcal{M}, \pi)\) to \((\mathcal{M}', \pi')\) is a graded manifold morphism \(\psi : \mathcal{M} \to \mathcal{M}'\) such that

\[
\psi^* \{f_1, \ldots, f_k\}_{\pi'} = \{\psi^* f_1, \ldots, \psi^* f_k\}_{\pi}
\]

for \(f_1, \ldots, f_k \in \mathcal{C}^\infty(\mathcal{M}')\).

Equivalently, \(\pi \sim \pi'\).

Weak morphisms??
**hPoisson-Lie groups**

**Definition**
A *hPoisson-Lie group* is a graded Lie group $G$ equipped with a hPoisson structure such that the multiplication map $\mu : G \times G \to G$ is a hPoisson morphism.

**Examples**
Poisson-Lie groups, $Q$-groups/dg-groups,...

**Definition**
A *hPoisson-Lie group* is a graded Lie group $G$ where $T^*[1]G$ is equipped with a multiplicative homological symplectic vector field, or equivalently, a degree 2 multiplicative function $\phi$ such that $[\phi, \phi] = 0$.

“Multiplicative” refers to the groupoid structure $T^*[1]G \rightrightarrows g^*[1]$. 
Homotopy Lie bialgebras

A multiplicative homological symplectic vector field $d_\phi$ on $T^*[1]\mathcal{G} \Rightarrow g^*[1]$ lives over a homological Poisson vector field $\hat{d}_\phi$ on $g^*[1]$, which can be thought of as a differential on $\mathcal{C}^\infty(g^*[1]) = S(g[-1])$ (think $\wedge g$).

$\hat{d}_\phi$ Poisson $\iff$ derivation of the Schouten-Lie bracket.

Definition

A \textit{homotopy Lie bialgebra} is a graded Lie algebra $g$ equipped with a differential $\delta$ on $S(g[-1])$ that is a derivation of symmetric product and the Schouten-Lie bracket.

- If $\delta$ is linear, then $g$ is a DGLA (= Lie $Q$-algebra).
- If $\delta$ is quadratic, then $g$ is a graded Lie bialgebra.
- In general, the derivation property expresses a compatibility between a graded Lie algebra structure on $g$ and an $L_\infty$-algebra structure on $g^*$. 
Let $M$ be a hPoisson manifold, and let $G$ be a hPoisson-Lie group.

**Definition**
An action $\sigma: M \times G \to M$ is *hPoisson* if $\sigma$ is a hPoisson morphism.

Infinitesimal version: Let $g$ be a homotopy Lie bialgebra.

**Definition**
An action $\rho: g \to \mathfrak{X}(M)$ is a *homotopy Lie bialgebra action* if the extension $\hat{\rho}: S(g[-1]) \to \mathfrak{X}^\bullet(M)$ respects differentials.

**Lemma**
_Suppose that $G$ has a free and proper hPoisson action on $M$. Then the quotient $M/G$ inherits a hPoisson structure._
Hamiltonian actions

Let $S$ be a degree 1 symplectic $Q$-manifold. Let $(G, \phi)$ be a connected hPoisson-Lie group with a Hamiltonian action on $S$ with moment map $\mu : S \rightarrow \mathfrak{g}^*[1]$.

Recall that $\mathfrak{g}^*[1]$ has a homological vector field $\hat{d}_\phi$.

**Definition**
The action is called $Q$-Hamiltonian if $\mu$ is a $Q$-manifold morphism. Equivalently, $\mu^* : S(\mathfrak{g}[-1]) \rightarrow C^\infty(S)$ respects differentials.

**Theorem**
*If $G$ is flat and the action is $Q$-Hamiltonian (+ regular value, etc.), then the homological vector field on $S$ descends to the quotient $\mu^{-1}(0)/G$.***

Nonflat $\iff$ reduction at nonzero values?
Let $\mathcal{M}$ be a hPoisson manifold, and let $\mathcal{G}$ be a flat hPoisson-Lie group with a free and proper hPoisson action on $\mathcal{M}$.

$\rightsquigarrow$ (shifted) cotangent lift action $\mathcal{G} \bowtie T^*[1]\mathcal{M}$.

**Theorem**

The cotangent lift action is $Q$-Hamiltonian, and the reduced symplectic $Q$-manifold is $T^*[1](\mathcal{M}/\mathcal{G})$.

**Example**

If $M$ is a Poisson manifold and $G$ is a Poisson-Lie group with a free and proper Poisson action on $M$, then the Poisson quotient $M/G$ can be interpreted as arising from the “$Q$-symplectic quotient” $T^*[1]M//G$. 
Higher hPoisson structures

Let $\mathcal{M}$ be a graded manifold.

**Definition**
A *degree $n$ hPoisson structure* on $\mathcal{M}$ is a degree $n + 1$ function $\pi$ on $T^*[n]\mathcal{M}$ such that $[\pi, \pi] = 0$.

degree $n$ hPoisson-Lie groups can do $Q$-symplectic reduction on degree $n$ symplectic $Q$-manifolds.

**Example**
Bursztyn-Cavalcanti-Gualtieri notion of “extended action with moment map” for reduction of Courant algebroids. (In this case, the deg. 2 homotopy Lie bialgebra is a DGLA.)
The quadratic case

Example

Quadratic deg. 2 homotopy Lie bialgebras correspond to "matched pairs" of Lie algebras.

Interesting example of Courant reduction by "matched pair action"?
Thanks.