

Notes on Eigenvalues, eigenvectors, and diagonalization

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These notes give an introduction to eigenvalues, eigenvectors, and diagonalization, with an emphasis on the application to solving systems of differential equations. The key point is that, when you solve such systems, the linear algebra really does the “heavy lifting”, and calculus is only necessary for one easy step of the process.

The first two sections focus on the big picture of why you would care about eigenvalues, eigenvectors, and diagonalization, and Section 3 explains how to actually find eigenvalues and eigenvectors. Depending on how you like to learn math, you might prefer to read Section 3 first, and then come back to the first two sections afterward.

1 Big picture: Systems of linear differential equations

1.1 Describing systems of linear differential equations in vector form

The main motivation for eigenvalues and eigenvectors is their application in solving systems of linear differential equations. An example of a system of linear differential equations is

$$\begin{aligned}x_1' &= 2x_1 + 3x_2, \\x_2' &= x_1 + 4x_2.\end{aligned}$$

In this type of equation, x_1 and x_2 are both supposed to be functions of t , and you’re trying to find what x_1 and x_2 can be, given that they need to satisfy the above equations.

What makes this kind of system hard to solve is the fact that the variables x_1 and x_2 interact — the derivative of x_1 depends on x_2 , and the derivative of x_2 depends on x_1 — so there’s a feedback mechanism that’s difficult to resolve. But we will see that linear algebra can be used to reduce the system to a simpler one, where the feedback mechanism doesn’t exist.

First, observe that we can express the above system in vector form. If we let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix},$$

then the system of equations can be written as

$$\mathbf{x}' = A\mathbf{x}.$$

More generally, any system of linear differential equations (at least those that are *homogeneous*, i.e. that don’t contain any constant terms) can be written in the form $\mathbf{x}' = A\mathbf{x}$ for some square matrix A .

1.2 Similar matrices

Our approach to solving systems of linear differential equations will utilize the concept of *matrix similarity*:

Definition. Two square matrices A and B are called *similar* if there is an invertible matrix P such that $P^{-1}AP = B$ (or, equivalently, $A = PBP^{-1}$).

There is a sense in which you could say that two similar matrices are “essentially the same.” Specifically, since P is invertible, you could view it as just being a “change of perspective,” or more formally, a change of basis. If $P^{-1}AP = B$, then that means that doing the transformation B is the same thing as doing P (changing perspective), then doing A , and then doing P^{-1} (changing back to the original perspective).

Note that, if you know that P is invertible, then it’s good enough to check that $AP = PB$. It is helpful to think of the following diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xleftarrow{B} & \mathbb{R}^n \\ P \downarrow & & \downarrow P \\ \mathbb{R}^n & \xleftarrow{A} & \mathbb{R}^n \end{array}$$

The equation $P^{-1}AP = B$ means that you could start in the upper right-hand corner and use B to go to the upper left-hand corner, or you could use P to go down, A to go left, and then P^{-1} to go up, and the result should be the same either way. Alternatively, the equation $AP = PB$ means that both ways of going from the upper right corner to the lower left corner should produce the same result.

I’ll emphasize that similarity is a very strong relationship, in contrast to row-equivalence, which is a very weak relationship. If you pick two matrices at random they will most likely *not* be similar, whereas they most likely *will* be row-equivalent. If two matrices are similar, they will have many properties in common. For example, they will have the same determinant, since

$$\det A = \det P \det B \det P^{-1} = \det B$$

(remember $\det P^{-1} = 1/\det P$, so you can cancel.) We will later discover more common properties.

1.3 Similarity and systems of linear differential equations

Let’s suppose you want to solve the system $\mathbf{x}' = A\mathbf{x}$ for some matrix A . And suppose that A is similar to another matrix $B = P^{-1}AP$ which is somehow “nicer” (we’ll soon clarify what “nicer” might mean).

You can view the invertible matrix P as giving a change of variables

$$\mathbf{x} = P\mathbf{y}, \text{ or equivalently } \mathbf{y} = P^{-1}\mathbf{x}.$$

This is actually just a fancy case of a u -substitution (like in calculus), which we can use to rewrite the equation $\mathbf{x}' = A\mathbf{x}$ in terms of \mathbf{y} .

Since P doesn’t depend on t , we can treat it as a constant and “pull it out” when we take the derivative of \mathbf{x} :

$$\mathbf{x}' = P\mathbf{y}'.$$

Now we can substitute $\mathbf{x} = P\mathbf{y}$ and $\mathbf{x}' = P\mathbf{y}'$ into the equation $\mathbf{x}' = A\mathbf{x}$, to get

$$P\mathbf{y}' = AP\mathbf{y}.$$

If we multiply both sides of the equation by P^{-1} , we get

$$\mathbf{y}' = P^{-1}A\mathbf{P}\mathbf{y} = B\mathbf{y}.$$

The conclusion is that, if A is similar to B , then the change of variables $\mathbf{x} = \mathbf{P}\mathbf{y}$ turns the system $\mathbf{x}' = A\mathbf{x}$ into the system $\mathbf{y}' = B\mathbf{y}$. This means we can (potentially) use the following strategy to solve the equation $\mathbf{x}' = A\mathbf{x}$:

1. First solve the equation $\mathbf{y}' = B\mathbf{y}$.
2. Then multiply those solutions by \mathbf{P} to solve for \mathbf{x} .

Of course, this strategy is only useful if it's somehow easier to solve the equation involving B than it is to solve the equation involving A . So we arrive at the following question:

Given a square matrix A , how can we find a similar matrix B for which the corresponding system of equations is easier to solve?

Everything that we will do from here on will be motivated by trying to answer this question.

2 Diagonalization

2.1 Diagonal matrices and decoupled systems

The easiest systems to solve are the ones where the matrix is *diagonal*, meaning that the only nonzero entries are along the diagonal. In the 2×2 case, such a matrix is of the form

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

and the corresponding system of differential equations $\mathbf{y}' = D\mathbf{y}$ is

$$\begin{aligned} y_1' &= \lambda_1 y_1, \\ y_2' &= \lambda_2 y_2. \end{aligned}$$

What's great about this system is that it is *decoupled* — there's an equation for y_1 and an equation for y_2 , and the two variables don't interact with each other. In fact, these equations are simple exponential growth equations, and the solutions are

$$\begin{aligned} y_1 &= c_1 e^{\lambda_1 t}, \\ y_2 &= c_2 e^{\lambda_2 t}. \end{aligned}$$

(Note: This is the only step in the whole process that requires calculus.)

2.2 Diagonalization

Suppose that you are given an arbitrary $n \times n$ matrix A . The discussions in Sections 1.3 and 2.1 motivate the following question: **Can we find a diagonal matrix D that is similar to A ?**

Let's try to figure out what this would involve. We'll use the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xleftarrow{D} & \mathbb{R}^n \\ P \downarrow & & \downarrow P \\ \mathbb{R}^n & \xleftarrow{A} & \mathbb{R}^n \end{array}$$

If we start with \mathbf{e}_1 in the upper right-hand corner, then we could multiply by D and get

$$D\mathbf{e}_1 = \lambda_1\mathbf{e}_1.$$

Alternatively, we could multiply by P . We don't know what P is yet, so let's just say that the columns of P are some yet-undetermined vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Then, when we multiply \mathbf{e}_1 by P , we'll get the first column:

$$P\mathbf{e}_1 = \mathbf{v}_1.$$

Also, since λ_1 is a scalar, we can pull it out when computing PDE_1 :

$$PDE_1 = P(\lambda_1\mathbf{e}_1) = \lambda_1P(\mathbf{e}_1) = \lambda_1\mathbf{v}_1.$$

This is enough information to fill in all four corners of the diagram:

$$\begin{array}{ccc} \lambda_1\mathbf{e}_1 & \xleftarrow{D} & \mathbf{e}_1 \\ P \downarrow & & \downarrow P \\ \lambda_1\mathbf{v}_1 = A\mathbf{v}_1 & \xleftarrow{A} & \mathbf{v}_1 \end{array}$$

The key point here is that **the only way to make the diagram “work”** is if $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$. This equation is called the *eigenvalue equation*:

Definition. An *eigenvector* of a square matrix A is a nonzero vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ . We then say that λ is the *eigenvalue* associated to \mathbf{v} .

A more thorough discussion of eigenvalues and eigenvectors is in the next section. But for now, let's just use the concept to complete the discussion of diagonalizability.

Using the “eigen” terminology, we would say that, to make the above diagram “work”, \mathbf{v}_1 needs to be an eigenvector, and λ_1 needs to be the corresponding eigenvalue. And if we started with \mathbf{e}_2 in the upper right-hand corner, we could conclude that \mathbf{v}_2 needs to be an eigenvector, etc.

We need to have n eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ to form the matrix P , and since P needs to be invertible, these eigenvectors need to form a basis. But that's all we need. This solves the problem (in theory, at least):

Theorem 2.1. If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of eigenvectors for an $n \times n$ matrix A , then you can form the matrix $P = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$, and it will **automatically** be true that $P^{-1}AP = D$, where D is the diagonal matrix whose entries are the corresponding eigenvalues:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

I'll emphasize that this theorem doesn't necessarily give a positive answer to the question at the beginning of the section. The issue is that, depending on what A is, it may or may not be possible to find a basis of eigenvectors. In later sections, we'll go through all the different possibilities for what can happen.

Terminology: The process of finding the P and the D such that $P^{-1}AP = D$ is called *diagonalization*. If it is possible to diagonalize A (in other words, if there exists a basis of eigenvectors), then you would say that A is *diagonalizable*.

3 Eigenvalues and eigenvectors

3.1 Geometric meaning of the eigenvalue equation

Recall that, if A is a square matrix, then an *eigenvector* of A is a nonzero vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ . In this case, λ is called an *eigenvalue*. Note that λ is allowed to be 0, but \mathbf{v} isn't.

Geometrically, the eigenvalue equation says that, if \mathbf{v} is an eigenvector, then when you multiply it by A , the result lies on the line spanned by \mathbf{v} . It might be that $A\mathbf{v}$ is longer than \mathbf{v} or shorter than \mathbf{v} . It might be that $A\mathbf{v}$ is pointing in exactly the opposite direction to \mathbf{v} (in this case λ is negative), and it might even be that $A\mathbf{v}$ is the zero vector (in this case $\lambda = 0$).

3.2 How to solve for eigenvalues and eigenvectors

To solve the eigenvalue equation, it is helpful to rewrite it in the following way:

$$(A - \lambda I)\mathbf{v} = 0.$$

(To get here, we moved the $\lambda\mathbf{v}$ term over to the left and used the fact that λI is the matrix that "implements" scalar multiplication by λ .) **This equation simply says that \mathbf{v} is in the nullspace of $A - \lambda I$.**

One way to know whether or not there exists a nonzero vector in the nullspace of a square matrix is to take its determinant; if the determinant is 0, then there does exist a nonzero vector in the nullspace, and if the determinant is nonzero, then the nullspace is trivial. Applying this principle to the equation $(A - \lambda I)\mathbf{v} = 0$, we conclude that a nonzero solution \mathbf{v} exists exactly when $\det(A - \lambda I) = 0$.

This fact gives us a relatively straightforward (but sometimes tedious!) procedure to find all the eigenvalues and eigenvectors of A :

1. Compute $\det(A - \lambda I)$, with λ being treated as undetermined. The result will be a polynomial in the variable λ . This polynomial is called the *characteristic polynomial* of A .
2. Factor the characteristic polynomial. The roots are the eigenvalues. (They are precisely the values of λ for which the eigenvalue equation has a nonzero solution \mathbf{v} .)
3. For each eigenvalue λ , compute the nullspace of $A - \lambda I$. The nonzero vectors in the nullspace are the eigenvectors for eigenvalue λ , and the entire nullspace is referred to as the *eigenspace* for λ . Usually, you will want to find a basis for every eigenspace.

3.3 Let's go crazy with examples

There are *a lot* of examples below, because there are a lot of subtle issues that can arise in the details. I highly recommend that you go over each example carefully, because each one illustrates a different issue. The best thing would be for you to follow along by doing the computations yourself.

Example 3.1. Problem: Find the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{bmatrix} = (2 - \lambda)(4 - \lambda) - 3.$$

At this point, you would usually expand the parentheses and refactor. (There are some cases where other tricks can be employed; I'll demonstrate these in later examples.) In this case, you get

$$8 - 6\lambda + \lambda^2 - 3 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1),$$

so the eigenvalues are $\lambda = 5$ and $\lambda = 1$.

The next step is to solve for the eigenvectors. For $\lambda = 5$, we have

$$A - 5I = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}.$$

Note that both rows are multiples of each other — this is to be expected in the 2×2 case, because we picked λ to make the matrix “degenerate”.

The eigenspace is just the space of solutions to the homogeneous equation for this matrix. I'm going to assume you're good at solving homogeneous equations by now, so I'll leave out the details for this step. The solution space (hence the eigenspace for $\lambda = 5$) is the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly, for $\lambda = 1$, we have

$$A - 1I = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix},$$

and the eigenspace is the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. ◇

Example 3.2. Problem: Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 3 \\ 3 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2 - 9.$$

It's still fine to expand the parentheses and refactor if you want. But here is a slightly more fancy approach you can use in this case:

$$(2 - \lambda)^2 - 9 = 0,$$

$$(2 - \lambda)^2 = 9,$$

$$2 - \lambda = \pm 3,$$

so the eigenvalues are $\lambda = 2 \pm 3$ (in other words, $\lambda = 5$ and $\lambda = -1$).

For $\lambda = 5$, we have

$$A - 5I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix},$$

and the eigenspace is the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = -1$, we have

$$A - (-1)I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix},$$

and the eigenspace is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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Example 3.3. Problem: Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 3 \\ 3 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - 9.$$

If you expand everything, you get

$$\lambda^2 - 5\lambda - 5.$$

Remember that if you're having trouble factoring, you can always rely on the quadratic formula, which tells you that the roots are

$$\lambda = \frac{5 \pm \sqrt{25 + 20}}{2} = \frac{5 \pm \sqrt{45}}{2} = \frac{5 \pm 3\sqrt{5}}{2},$$

which unfortunately can't be simplified any further.

The process for finding the corresponding eigenspaces is the same as the other examples, although the square roots will make the numbers a bit ugly. (I wouldn't ask you to do it by hand, but it's good for you to know that it can be done, in principle.)

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Example 3.4. Problem: Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & -1 \\ 1 & -\lambda \end{bmatrix} = -\lambda(2 - \lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

so there's only one eigenvalue $\lambda = 1$.

Then

$$A - I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

and the eigenspace is the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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Example 3.5. Problem: Find the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & -5 \\ 1 & -1 - \lambda \end{bmatrix} = (3 - \lambda)(-1 - \lambda) + 5 = \lambda^2 - 2\lambda + 2.$$

This is another tricky one, so we can use the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{4 - 8}}{2} = \odot.$$

Uh oh! There's a negative number under the square root, so this quadratic doesn't have any real roots! That means that A doesn't have any real eigenvalues, and therefore it doesn't have any real eigenvectors. ◇

And now for some 3×3 examples. In general, the 3×3 case is hard to do by hand (mostly because it can be difficult to factor cubic polynomials by hand), but the following examples illustrate some cases where it's doable.

Example 3.6. Problem: Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: This matrix is lower-triangular, and so is $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0 \\ 1 & 2 & 3 - \lambda \end{bmatrix}.$$

The determinant of a triangular matrix is just the product of the diagonal entries, which in this case is $(2 - \lambda)(4 - \lambda)(3 - \lambda)$. So the eigenvalues are $\lambda = 2, 4, 3$. Note that the eigenvalues are just the entries along the diagonal — this is always the case for triangular matrices, and it's fine to simply write “ A is triangular, so the eigenvalues are the diagonal entries 2, 4, 3.”

For $\lambda = 2$, we have

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

Typically, you will need to row-reduce in the 3×3 case. The reduced form is

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so you end up with the equations

$$\begin{aligned}x_1 &= -x_2, \\x_2 &\text{ free,} \\x_3 &= 0,\end{aligned}$$

and the eigenspace is

$$\left\{ \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{Span} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 4$, we have

$$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}.$$

The reduced form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

so you end up with the equations

$$\begin{aligned}x_1 &= 0, \\x_2 &= \frac{1}{2}x_3, \\x_3 &\text{ free,}\end{aligned}$$

and the eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} \right\} = \left\{ x_3 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\} = \text{Span} \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \text{Span} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

(Note that in the last step I just multiplied the vector by 2 so there wouldn't be any fractions. You're allowed to do this because it doesn't change the span.)

For $\lambda = 3$, we have

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

The reduced form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so you end up with the equations

$$\begin{aligned}x_1 &= 0, \\x_2 &= 0, \\x_3 &\text{ free,}\end{aligned}$$

and the eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \right\} = \left\{ x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Span} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

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Example 3.7. Problem: Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solution: The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 3 & 0 \\ 1 & 4 - \lambda & 0 \\ 0 & 0 & 5 - \lambda \end{bmatrix}.$$

Expand over the last row or last column to get

$$(5 - \lambda) \det \begin{bmatrix} 2 - \lambda & 3 \\ 1 & 4 - \lambda \end{bmatrix} = (5 - \lambda) [(2 - \lambda)(4 - \lambda) - 3].$$

This polynomial is already partially factored, so just leave the $(5 - \lambda)$ out there and factor the other part. That part is just like in Example 3.1, and you'll end up with $(5 - \lambda)^2(1 - \lambda)$, so the eigenvalues are $\lambda = 5$ and $\lambda = 1$. It's worth remembering that 5 appears as a "double root"; this will turn out to have some significance.

For $\lambda = 5$, we have

$$A - 5I = \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The reduced form is

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so you end up with the equations

$$\begin{aligned} x_1 &= x_2, \\ x_2, x_3 &\text{ free,} \end{aligned}$$

and the eigenspace is

$$\left\{ \begin{bmatrix} x_2 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the eigenspace is 2-dimensional here. This isn't something that always happens, but it's what you hope for when you've got a double root.

For $\lambda = 1$, we have

$$A - 1I = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

The reduced form is

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so you end up with the equations

$$\begin{aligned} x_1 &= -3x_2, \\ x_2 &\text{ free,} \\ x_3 &= 0, \end{aligned}$$

and the eigenspace is

$$\left\{ \begin{bmatrix} -3x_2 \\ x_2 \\ 0 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{Span} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

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3.4 Similar matrices and the characteristic polynomial

If you are given two square matrices A and B , one question you could ask is whether A and B are similar or not. (Remember that similar matrices essentially describe the same operation up to a change of basis, so two different people attempting to describe the same system might need to answer this question when they compare their work.)

To prove that A and B are similar, you could just find a P such that $P^{-1}AP = B$, and that would do it. But what if you thought A and B actually *aren't* similar? The issue is that it isn't good enough to say "I tried to find a P and couldn't do it." **You need to somehow prove that it isn't possible to find P .** One way to do this is to look for things called "invariants" — quantities that similar matrices have in common. Here are some invariants that you already know about:

- Rank: If A and B are similar, then $\text{rank}(A) = \text{rank}(B)$. So if A and B have different ranks, then you would know that they are not similar. Warning: Similar matrices *can* have different row-reduced forms, but they should have the same number of leading variables.
- Nullity: If A and B are similar, then $\text{nullity}(A) = \text{nullity}(B)$. (Although, because of the Rank-Nullity Theorem, this essentially provides the same information as the rank.)
- Determinant: If A and B are similar, then $\det A = \det B$ (The explanation was in Section 1.2). So, if A and B have different determinants, then you would know that they are not similar.

But the above invariants aren't all that sensitive, in the sense that there are a lot of examples of matrices that have the same rank, nullity, and determinant, but still aren't similar. For example,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

all have determinant 1, rank 2, and nullity 0, but we will soon see that none of them is similar to any other.

The following theorem provides a much more powerful invariant.

Theorem 3.8. *If A and B are similar, then A and B have the same characteristic polynomial.*

Proof. I claim that, if A and B are similar, then $A - \lambda I$ and $B - \lambda I$ are similar as well. Specifically, if $P^{-1}AP = B$, then

$$P^{-1}(A - \lambda I)P = P^{-1}AP - \lambda P^{-1}P = B - \lambda I.$$

We know that similar matrices have the same determinant, so $\det(A - \lambda I) = \det(B - \lambda I)$. In other words, the characteristic polynomials of A and B are equal. \square

Corollary 3.9. *If A and B are similar, then A and B have the same eigenvalues.*

Warning: Similar matrices don't necessarily have the same *eigenvectors*!

For the three matrices above, the characteristic polynomials are, respectively,

$$\lambda^2 - 2\lambda + 1, \lambda^2 - 4\lambda + 1, \lambda^2 - 5\lambda + 1.$$

The fact that they all have different characteristic polynomials qualifies as proof that they are not similar to each other.

Warning: The characteristic polynomial still isn't a perfect invariant. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the same characteristic polynomial as the identity matrix, even though it isn't similar to the identity matrix.

4 More about diagonalization

4.1 When is a matrix diagonalizable?

Given a square matrix A , you could imagine two questions related to its diagonalizability. One would simply ask, "Is A diagonalizable?" The other would actually ask you to diagonalize A (that is, find the P and D), assuming that A is diagonalizable. In many cases, it's possible to determine whether A is diagonalizable without doing all the work required to actually diagonalize A . One of the facts that makes this possible is the following theorem:

Theorem 4.1. *Any set of eigenvectors that all correspond to different eigenvalues will automatically be linearly independent.*

A simple conclusion you can draw from this theorem is the following:

Corollary 4.2. *If an $n \times n$ matrix has n different eigenvalues, then you know for sure that it is diagonalizable.*

Proof. Each eigenvalue has at least one associated eigenvector (otherwise it wouldn't be an eigenvalue!). So, if you had n different eigenvalues, then you could find one eigenvector for each eigenvalue. They will automatically be linearly independent because of Theorem 4.1, so they form a basis of eigenvectors. \square

The point here is that *you might not need to find the eigenvectors*, if you are only asked to determine diagonalizability. If your characteristic polynomial fully factors without any double roots, then the matrix is diagonalizable for sure.

Example 4.3. Problem: Is the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

diagonalizable?

Solution: This is the same matrix in Example 3.1, so I won't repeat the work here. But the point is that this question doesn't require you to do all the work we did in Example 3.1. As soon as you get to the point where you know that the eigenvalues are $\lambda = 5$ and 1, then you can say "Yes, it is diagonalizable, and we know this because there are two different eigenvalues". \diamond

Similarly, you can say that the matrices in Examples 3.2 and 3.3 are diagonalizable because they both have two different eigenvalues. (Note that in Example 3.3 we didn't bother finding any eigenvectors, but we didn't need to do so to determine diagonalizability.) Also, because the matrix in Example 3.6 is lower-triangular, you can quickly see that the eigenvalues are $\lambda = 2, 4, 3$, which are all different, and conclude that it's diagonalizable without doing any work at all. Here's a 4-dimensional example:

Example 4.4. Problem: Is the matrix

$$\begin{bmatrix} 1 & 4 & -2 & 6 \\ 0 & 4 & 6 & 2 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

diagonalizable?

Solution: It is upper-triangular, so its eigenvalues are the numbers along the diagonal: 1, 4, 0, 2. Since all four eigenvalues are different, you can conclude that it is diagonalizable. \diamond

What about the case where you don't have n different eigenvalues? This can actually happen in two different ways:

- **It's possible that the characteristic polynomial doesn't fully factor**, like in Example 3.5. In this case, the matrix won't be diagonalizable, at least not in terms of real numbers. However, it may still be possible to diagonalize it using *complex numbers*; we'll see how this works (at least in the 2×2 case) in Section 5. But for now, let's say that the answer is "No-with-an-asterisk".
- **It's possible that the characteristic polynomial has a repeated root**, like in Examples 3.4 and 3.7. These cases are more tricky. In order for such a matrix to be diagonalizable, any eigenvalue that's a double root should have a 2-dimensional eigenspace, any eigenvalue that's a triple root should have a 3-dimensional eigenspace, etc. So, in general, you will still have to do some work toward computing eigenspaces, but you can restrict your attention to the eigenspaces for repeated-root eigenvalues. Some examples are below.

Example 4.5. Problem: Is the matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

diagonalizable?

Solution: This is the matrix from Example 3.4. As we found there, it has a double root $\lambda = 1$, and the corresponding eigenspace is only 1-dimensional. So this matrix is not diagonalizable. In Section 6, we'll learn about the "next-best thing" to diagonalization for this one. \diamond

Example 4.6. Problem: Is the matrix

$$\begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

diagonalizable?

Solution: This is the matrix from Example 3.7. As we saw there, $\lambda = 5$ is a double root, but luckily the corresponding eigenspace is 2-dimensional. So we can conclude that this matrix is diagonalizable. Note that you don't need to compute the $\lambda = 1$ eigenspace in order to answer this question. \diamond

Example 4.7. Problem: Is the matrix

$$\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

diagonalizable?

Solution: The characteristic polynomial factors as $(\lambda - 3)^2(\lambda - 4)$ (I'll leave the details of that calculation to you). Because $\lambda = 3$ is a double root, we need to check whether or not the corresponding eigenspace is 2-dimensional.

We have

$$A - 3I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix},$$

which row-reduces to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is only one free variable, so the eigenspace is only 1-dimensional. We can conclude that the matrix is not diagonalizable. \diamond

4.2 Examples of diagonalization

Let's look again at the examples from Section 3.3. For each of the matrices, we'll answer the following question:

Problem: If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

We already did the necessary work to answer this question in Section 3.3, so it's just a matter of presenting it in the appropriate way. The idea is that P is made up out of a basis of eigenvectors, and D is made up out of the corresponding eigenvalues.

- Example 3.1:

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Example 3.2:

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

- (Example 3.3 is skipped because who wants to compute the eigenvectors for that)

- Example 3.4: Not diagonalizable
- Example 3.5: Not diagonalizable (at least in terms of real numbers)
- Example 3.6:

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

- Example 3.7:

$$P = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4.3 Using diagonalization to solve systems of differential equations

Finally, we're ready to put the strategy outlined in Section 1.3 into action. If a matrix A is diagonalizable, then you can solve the system $\mathbf{x}' = A\mathbf{x}$ with the following steps:

1. Diagonalize A . That is, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Doing this involves the following substeps:
 - (a) Find the eigenvalues of A .
 - (b) For each eigenvalue, find the eigenspace, and express it as the span of a set of linearly independent vectors. (The number of vectors in the set should be equal to the multiplicity of the eigenvalue, which more often than not is just one.)
 - (c) The matrix P is made up out of all of the spanning vectors you found for the eigenspaces, and D is made up out of the corresponding eigenvalues.
2. Solve the system $\mathbf{y}' = D\mathbf{y}$. This is easy to do because the system is decoupled.
3. The solutions to the original system are given by $\mathbf{x} = P\mathbf{y}$.

Example 4.8. Problem: Find all solutions to the system of differential equations

$$\begin{aligned} x_1' &= 2x_1 + 3x_2, \\ x_2' &= x_1 + 4x_2. \end{aligned}$$

Solution: In this case, A is the matrix in Example 3.1, so we've already done step 1. We got

$$P = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

For step 2, we consider the system $\mathbf{y}' = D\mathbf{y}$, or

$$\begin{aligned} y_1' &= 5y_1, \\ y_2' &= y_2. \end{aligned}$$

The solutions are simply $y_1 = c_1 e^{5t}$ and $y_2 = c_2 e^t$ (remember that in general, each variable is allowed to have a different constant), which can be written in vector form as

$$\mathbf{y} = \begin{bmatrix} c_1 e^{5t} \\ c_2 e^t \end{bmatrix}.$$

For step 3, we just need to multiply by P to get \mathbf{x} :

$$\mathbf{x} = P\mathbf{y} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{5t} \\ c_2 e^t \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} - 3c_2 e^t \\ c_1 e^{5t} + c_2 e^t \end{bmatrix}$$

It's common to split the solution into the parts involving c_1 and c_2 , and to pull out the exponential scalars, like this:

$$\mathbf{x} = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

And that's the final answer. ◇

Observe that the the vectors in the final answer of Example 4.8 are the eigenvectors. This is not a coincidence. Here's the general explanation. Suppose that A is a 2×2 matrix, and that \mathbf{v}_1 and \mathbf{v}_2 are linearly independent eigenvectors with eigenvalues λ_1 and λ_2 , so that

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2], \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

The solutions to $\mathbf{y}' = D\mathbf{y}$ can be written as

$$\mathbf{y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} = c_1 e^{\lambda_1 t} \mathbf{e}_1 + c_2 e^{\lambda_2 t} \mathbf{e}_2.$$

Then, because P sends \mathbf{e}_1 to \mathbf{v}_1 and \mathbf{e}_2 to \mathbf{v}_2 , the solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x} = P\mathbf{y} = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

If you've solved linear systems of equations in any of your other classes, you were probably just taught to use this last formula as the "rule" for solving a system. Now you know where that rule comes from.

5 Complex eigenvalues

5.1 Diagonalization over the complex numbers

As we observed in Section 4.1, one way that a matrix might fail to be diagonalizable is that its characteristic polynomial might not fully factor. In the 2×2 case, this happens when the quadratic formula gives you a negative quantity under the square root. We saw that this happened in Example 3.5.

One way to get around this problem is to expand your universe to include *complex numbers*. This is a more dramatic shift than you might realize — it means that **everything should be allowed to be complex**. The scalars can be complex numbers instead of real numbers, the vectors can have complex entries (so the space is \mathbb{C}^n instead of \mathbb{R}^n), and the matrices can have complex entries also. Conceptually, this is a big deal.

But from a practical point of view, it's actually not that much of a change. Everything works exactly the same way as before, except that all the arithmetic involves complex numbers.

Example 5.1. Problem: Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}.$$

Using complex numbers, find the eigenvalues and corresponding eigenspaces of A .

Solution: The characteristic polynomial is

$$(1 - \lambda)(3 - \lambda) + 2 = \lambda^2 - 4\lambda + 5.$$

Since $\sqrt{-4} = 2i$, the quadratic formula tells us that the roots are

$$\lambda = \frac{4 \pm \sqrt{16 - 20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Again, the eigenspaces are the nullspaces of $A - \lambda I$. For $\lambda = 2 + i$, we get

$$A - (2 + i)I = \begin{bmatrix} -1 - i & -2 \\ 1 & 1 - i \end{bmatrix}.$$

Now, (just like in the case of real eigenvalues) there should be a free variable here, which means the two rows should be multiples of each other. And they are, but it's less obvious than it is with real numbers (in this case, the first row is $-1 - i$ times the second row). But as long as you're sure you found your eigenvalues correctly, it's safe to assume that the rows are equivalent, and you can take either one of them and use the corresponding equation to determine the eigenspace. In this case, the second row is a little bit nicer, because it has the 1 in it. Its corresponding equation is

$$x_1 + (1 - i)x_2 = 0,$$

so

$$x_1 = (-1 + i)x_2$$

and x_2 is free, and the eigenspace is

$$\left\{ \begin{bmatrix} (-1 + i)x_2 \\ x_2 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} \right\} = \text{Span} \begin{bmatrix} -1 + i \\ 1 \end{bmatrix}.$$

In principle, the next step would be to find the eigenspace for the other eigenvalue $\lambda = 2 - i$. But actually, we can get this without doing any more work (see Theorem 5.2 below); it is simply the conjugate of the other eigenspace:

$$\text{Span} \begin{bmatrix} -1 - i \\ 1 \end{bmatrix}.$$

◇

The following theorem explains the last step in Example 5.1.

Theorem 5.2. *If a matrix has all real entries, then its eigenvalues and eigenvectors come in conjugate pairs. In other words, if \mathbf{x} is an eigenvector with eigenvalue λ , then its conjugate $\bar{\mathbf{x}}$ is an eigenvector with the conjugate eigenvalue $\bar{\lambda}$.*

Proof. If \mathbf{x} is an eigenvector with eigenvalue λ , then it satisfies the eigenvalue equation

$$A\mathbf{x} = \lambda\mathbf{x}.$$

If we take the conjugate of both sides of this equation, then everything gets conjugated:

$$\bar{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}.$$

Since all of the entries of A are real, it is actually equal to its own conjugate: $A = \bar{A}$. Therefore we see that

$$A\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}},$$

which says that $\bar{\mathbf{x}}$ is an eigenvector with eigenvalue $\bar{\lambda}$. □

This theorem makes diagonalization not too difficult in the case of complex roots. Here are a couple of examples that carry it out.

Example 5.3. Problem: Consider the matrix

$$A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}.$$

Using complex numbers, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution: This is the matrix from Example 3.5. We found there that the characteristic polynomial is $\lambda^2 - 2\lambda + 2$, so the quadratic formula gives

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

For $\lambda = 1 + i$, we have

$$A - (1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}.$$

Both rows should be equivalent, so let's use the equation for the second row because it has a 1:

$$x_1 + (-2-i)x_2 = 0,$$

$$x_1 = (2+i)x_2,$$

so the eigenspace is

$$\text{Span} \begin{bmatrix} 2+i \\ 1 \end{bmatrix}.$$

Because of Theorem 5.2, we don't have to do any further work to know that the eigenspace for the conjugate $\bar{\lambda} = 1 - i$ is

$$\text{Span} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}.$$

As usual, P is made up out of a basis of eigenvectors, and D is made up out of the corresponding eigenvalues:

$$P = \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

◇

Example 5.4. Problem: Diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

over the complex numbers.

Solution: The characteristic polynomial is

$$(3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

The roots are $\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$. For $\lambda = 1 + 2i$, we get

$$A - (1+2i)I = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}.$$

Both rows should be equivalent, so let's use the equation in the first row:

$$(2 - 2i)x_1 = 2x_2,$$

so we can take $x_1 = 2$ and $x_2 = (2 - 2i)$ (note the “cross-multiplication trick”). So the eigenspace is

$$\text{Span} \begin{bmatrix} 2 \\ 2 - 2i \end{bmatrix}.$$

For the conjugate eigenvalue $\bar{\lambda} = 1 - 2i$, the eigenspace is the conjugate

$$\text{Span} \begin{bmatrix} 2 \\ 2 + 2i \end{bmatrix},$$

so we can set

$$P = \begin{bmatrix} 2 & 2 \\ 2 - 2i & 2 + 2i \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{bmatrix}$$

and know that $P^{-1}AP = D$. ◇

5.2 Solving systems with complex eigenvalues

Suppose you wanted to solve the system $\mathbf{x}' = A\mathbf{x}$, where A is a 2×2 matrix with complex eigenvalues. In this case, the method described in Section 4.3 still works, but the problem is that you'll get *complex* solutions, even though you are most likely only looking for *real* solutions.

To illustrate what happens, let's use the matrix A from Example 5.3. In that case, the solutions to the equation $\mathbf{y}' = D\mathbf{y}$ are

$$\mathbf{y} = \begin{bmatrix} a_1 e^{(1+i)t} \\ a_2 e^{(1-i)t} \end{bmatrix} = a_1 e^{(1+i)t} \mathbf{e}_1 + a_2 e^{(1-i)t} \mathbf{e}_2,$$

so the solutions to the original equation are

$$\begin{aligned} \mathbf{x} &= P\mathbf{y} \\ &= \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix} \left(a_1 e^{(1+i)t} \mathbf{e}_1 + a_2 e^{(1-i)t} \mathbf{e}_2 \right) \\ &= a_1 e^{(1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} + a_2 e^{(1-i)t} \begin{bmatrix} 2-i \\ 1 \end{bmatrix}. \end{aligned}$$

The above is a linear combination of *conjugate* solutions, where the constants a_1 and a_2 are *complex* numbers. I'll skip the details, but with a little bit of complex arithmetic, you can conclude that the real solutions are linear combinations of the real and imaginary parts of these solutions. So we need to find the real and imaginary parts.

Using Euler's identity, we can rewrite the exponentials in terms of sine and cosine; in particular,

$$e^{(1+i)t} = e^t e^{it} = e^t (\cos t + i \sin t).$$

Then, if we focus on the part of the above solution that's multiplied by a_1 , we get

$$\begin{aligned} e^{(1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} &= e^t (\cos t + i \sin t) \begin{bmatrix} 2+i \\ 1 \end{bmatrix} \\ &= e^t \begin{bmatrix} 2 \cos t + 2i \sin t + i \cos t - \sin t \\ \cos t + i \sin t \end{bmatrix}. \end{aligned}$$

The real part of this is

$$e^t \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix},$$

and the imaginary part is

$$e^t \begin{bmatrix} 2 \sin t + \cos t \\ \sin t \end{bmatrix},$$

so the general solution consists of all linear combinations of the two:

$$\mathbf{x} = c_1 e^t \begin{bmatrix} 2 \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \sin t + \cos t \\ \sin t \end{bmatrix}.$$

6 Jordan form for 2×2 matrices

Even if a matrix has real eigenvalues, it might fail to be diagonalizable if the characteristic polynomial has repeated roots, and if the corresponding eigenspaces aren't big enough. In this case, you have to look for something else.

Recall that our motivation for diagonalization was the fact that the system $\mathbf{y}' = D\mathbf{y}$ is decoupled when D is diagonal, so it was easy to solve. At this point we need to search for other examples of “nice” matrices, for which the corresponding system is solvable. *Jordan matrices* provide an example of such a matrix.

In general, the theory of Jordan matrices is somewhat complicated, but we'll focus on the 2×2 case, which is more reasonable for us to do in an introductory course.

6.1 Jordan matrices

Definition: a 2×2 *Jordan matrix* is a matrix that looks like

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

for some number λ .

The corresponding system of differential equations $\mathbf{y}' = J\mathbf{y}$ is

$$\begin{aligned} y_1' &= \lambda y_1 + y_2, \\ y_2' &= \lambda y_2. \end{aligned}$$

This system isn't completely decoupled like it is for diagonal matrices, but there's still a clear path to solving it. First, solve the equation for y_2 :

$$y_2 = c_2 e^{\lambda t},$$

and then plug that into the equation for y_1 :

$$y_1' = \lambda y_1 + c_2 e^{\lambda t}.$$

This equation is a little more tricky, but it can be solved using a technique called “integrating factors”. I'll skip the details and tell you the answer:

$$y_1 = c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$

So, in vector form, the solutions are

$$\mathbf{y} = e^{\lambda t} \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix} = e^{\lambda t} (c_1 \mathbf{e}_1 + c_2(\mathbf{e}_1 + t\mathbf{e}_2)).$$

6.2 Jordanization

If A is a 2×2 matrix with only one eigenvalue λ , you could ask whether A is similar to the Jordan matrix

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Spoiler alert: it's going to turn out that the answer is always yes (except for the boring case where $A = \lambda J$). This is good news — it means that we can always solve the corresponding systems.

The process for finding the P in this case is surprisingly easy. Here is a step-by-step process:

1. Pick a nonzero vector \mathbf{v} that *isn't* an eigenvector. There are two ways to handle this. One is to figure out what the eigenvectors are, and then pick something different. The other way (which I would recommend) is to just pick *any* vector \mathbf{v} since, probabalistically, it's unlikely that a randomly chosen vector will be an eigenvector. And, in the unlucky case that you do pick an eigenvector, you would be able to figure it out in Step 2.
2. Let $\mathbf{u} = A\mathbf{v} - \lambda\mathbf{v}$. Important note: If you get $\mathbf{u} = 0$ here, then this means that \mathbf{v} is an eigenvector, and you should go back and pick a different \mathbf{v} .
3. Form P out of the vectors \mathbf{u} and \mathbf{v} :

$$P = [\mathbf{u} \quad \mathbf{v}].$$

And that's it! It will automatically be true that $P^{-1}AP = J$. This process doesn't have a good standard name (people might say it "puts a matrix in Jordan form"), but we'll call it *Jordanization*. The set of vectors $\{\mathbf{u}, \mathbf{v}\}$ is normally called a *Jordan basis*.

Of course, if you're surprised, skeptical, and/or mystified by the process, that's a good thing, since I just sprung these instructions on you without any explanation for why they work. I'll give the explanation below, but first let's demonstrate with an example:

Example 6.1. Problem: Consider the matrix

$$A = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{-1}AP$ is a Jordan matrix.

Solution: If you compute the eigenvalues of A , you'll find that it only has one eigenvalue $\lambda = 3$ (This is important, because if there were two different eigenvalues, you would diagonalize instead of Jordanize.). We'll go through the three steps described in Section 6.2.

Step 1 says to basically pick any vector at random. I'll pick $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because it's nice and simple.

Step 2 says to compute

$$\mathbf{u} = A\mathbf{v} - 3\mathbf{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

(At this point, if you got $\mathbf{u} = 0$, you would go back and pick a different \mathbf{v} .)

Step 3 says to form P out of \mathbf{u} and \mathbf{v} (I'll stress that the order in which you put the vectors matters here):

$$P = [\mathbf{u} \ \mathbf{v}] = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix},$$

and that's it. It will automatically be true that $P^{-1}AP = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. To check that it works, type this into WolframAlpha:

`{{2,1},{-2,0}}^(-1) * {{5,2},{-2,1}} * {{2,1},{-2,0}}`

I'll point out that, because we were able to pick \mathbf{v} almost completely randomly, the matrix P in this case is very much non-unique. For example, if we had picked $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then we would have gotten

$$\mathbf{u} = A\mathbf{v} - 3\mathbf{v} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

and

$$P = [\mathbf{u} \ \mathbf{v}] = \begin{bmatrix} 4 & 1 \\ -4 & 1 \end{bmatrix},$$

but it would still be true that $P^{-1}AP = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$, and that's the part that matters. ◇

6.3 Why does this strange process work?

To explain why the Jordanization process works, we need to figure out two things: First, how do we know that P is invertible? Second, how do we know that $P^{-1}AP = J$?

Invertibility of P is equivalent to linear independence of the vectors \mathbf{u} and \mathbf{v} . So, why should they be linearly independent? Well, if they weren't, then that would mean that one of them is a scalar multiple of the other. Then we would be able to write

$$\mathbf{u} = \mu\mathbf{v}$$

for some scalar μ . But, by definition, $\mathbf{u} = A\mathbf{v} - \lambda\mathbf{v}$, so this would mean

$$\begin{aligned} A\mathbf{v} - \lambda\mathbf{v} &= \mu\mathbf{v}, \\ A\mathbf{v} &= (\lambda + \mu)\mathbf{v}, \end{aligned}$$

which would mean that \mathbf{v} is an eigenvector with eigenvalue $\lambda + \mu$. This simply can't happen for $\mu \neq 0$ (since λ is the only eigenvalue). It could happen for $\mu = 0$, but that's exactly the case where $\mathbf{u} = 0$, which the process tells you to avoid. So we can conclude that, if you follow the process, the vectors \mathbf{u} and \mathbf{v} that you'll end up with will be linearly independent.

To explain why $P^{-1}AP = J$, let's return to the diagram that describes how similar matrices relate:

$$\begin{array}{ccc} \mathbb{R}^n & \xleftarrow{J} & \mathbb{R}^n \\ P \downarrow & & \downarrow P \\ \mathbb{R}^n & \xleftarrow{A} & \mathbb{R}^n \end{array}$$

Let's start with \mathbf{e}_2 in the upper right corner. If you go to the upper left corner, you get

$$J\mathbf{e}_2 = \mathbf{e}_1 + \lambda\mathbf{e}_2.$$

Alternatively, if you go to the lower right corner, you get $P\mathbf{e}_2$, which is \mathbf{v} (because \mathbf{v} is the second column of P). We can also compute

$$PJ\mathbf{e}_2 = P(\mathbf{e}_1 + \lambda\mathbf{e}_2) = \mathbf{u} + \lambda\mathbf{v}.$$

This is enough information to fill in all four corners of the diagram:

$$\begin{array}{ccc} \mathbf{e}_1 + \lambda\mathbf{e}_2 & \xleftarrow{J} & \mathbf{e}_2 \\ P \downarrow & & \downarrow P \\ \mathbf{u} + \lambda\mathbf{v} = A\mathbf{v} & \xleftarrow{A} & \mathbf{v} \end{array}$$

And you can see that, to make the diagram “work”, it's necessary that $\mathbf{u} + \lambda\mathbf{v} = A\mathbf{v}$. This is why the instructions tell you to set $\mathbf{u} = A\mathbf{v} - \lambda\mathbf{v}$.

But we also need to worry about what happens when we start with \mathbf{e}_1 in the upper right. In this case, the diagram is

$$\begin{array}{ccc} \lambda\mathbf{e}_1 & \xleftarrow{J} & \mathbf{e}_1 \\ P \downarrow & & \downarrow P \\ \lambda\mathbf{u} = A\mathbf{u} & \xleftarrow{A} & \mathbf{u} \end{array}$$

which tells us that \mathbf{u} needs to be an eigenvector. It turns out that this happens automatically. There's a theorem called the *Cayley-Hamilton Theorem* that says that (in this particular situation) $(A - \lambda I)^2 = 0$. This means that, for any vector \mathbf{v} ,

$$\begin{aligned} (A - \lambda I)^2\mathbf{v} &= 0, \\ (A - \lambda I)(A\mathbf{v} - \lambda\mathbf{v}) &= 0, \\ (A - \lambda I)\mathbf{u} &= 0, \end{aligned}$$

so \mathbf{u} is an eigenvector. So that's good, everything works the way it was supposed to, and we can conclude that $P^{-1}AP = J$.

A quick note on Cayley-Hamilton: The general statement of the Cayley-Hamilton Theorem is that a matrix “satisfies” its characteristic equation. In other words, if the characteristic polynomial of A is $\lambda^n + b_{n-1}\lambda^{n-1} + \dots + b_1\lambda + b_0$, then

$$A^n + b_{n-1}A^{n-1} + \dots + b_1A + b_0I = 0.$$

A general proof of Cayley-Hamilton would be better left for a more advanced course, but it is straightforward (albeit not very enlightening) to verify it in the 2×2 case. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the characteristic polynomial of A is $\lambda^2 - (a+d)\lambda + (ad - bc)$. By direct calculation, you can check that

$$A^2 - (a+d)A + (ad - bc)I = 0.$$

6.4 Using Jordanization to solve systems of differential equations

Example 6.2. Problem: Describe the solutions to the system of differential equations

$$\begin{aligned}x_1' &= 5x_1 + 2x_2, \\x_2' &= -2x_1 + x_2.\end{aligned}$$

Solution: The matrix for this system is the matrix A from Example 6.1. As we saw there, $P^{-1}AP = J$, where

$$P = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

Using the method described in Section 1.3, we can first solve the system $\mathbf{y}' = J\mathbf{y}$ and then set $\mathbf{x} = P\mathbf{y}$ to obtain solutions to the original system.

As we saw in Section 6.1, the solutions to $\mathbf{y}' = J\mathbf{y}$ are

$$\mathbf{y} = e^{3t} \begin{bmatrix} c_1 + c_2t \\ c_2 \end{bmatrix} = e^{3t} (c_1\mathbf{e}_1 + c_2(\mathbf{e}_1 + t\mathbf{e}_2)),$$

so the solutions to the original system are

$$\mathbf{x} = P\mathbf{y} = e^{3t} \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 + c_2t \\ c_2 \end{bmatrix} = e^{3t} \begin{bmatrix} 2c_1 + 2c_2t + c_2 \\ -2c_1 - 2c_2t \end{bmatrix} = c_1e^{3t} \begin{bmatrix} 2 \\ -2 \end{bmatrix} + c_2e^{3t} \begin{bmatrix} 2t + 1 \\ -2t \end{bmatrix}.$$

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In the general situation, the solutions to $\mathbf{y}' = J\mathbf{y}$ are

$$\mathbf{y} = e^{\lambda t} \begin{bmatrix} c_1 + c_2t \\ c_2 \end{bmatrix} = e^{\lambda t} (c_1\mathbf{e}_1 + c_2(\mathbf{e}_1 + t\mathbf{e}_2)),$$

and the solutions to the equation $\mathbf{x}' = A\mathbf{x}$ are

$$\mathbf{x} = P\mathbf{y} = e^{\lambda t} (c_1\mathbf{u} + c_2(\mathbf{u} + t\mathbf{v})).$$

If you've taken a course on differential equations, you might have learned a formula like this, but you probably didn't get a good explanation of where the formula came from. But now we've derived it from scratch.