

Errata: The Geometry of Spacetime

Corrections to 29 April 2015

I wish to express my thanks to David Griffel, University of Bristol, John Chiasson, University of Tennessee, Joseph North, George Mason University, Bill Rozzi, Alison Duren-Sutherland (Smith '02), Wayne Rossman, Kobe University, and members of the Kobe University undergraduate geometry seminar, Jan de Ruijter, David Berkowitz, José Carlos Santos, Grant Franks, Saburo Higuchi, Ryukoku University, Osamu Takeuchi, Lewis Robinson, Leonard Wong, and Richard De Simone for corrections and helpful comments about the text.

Corrections are marked in red, where possible.

Note: Errors that were found early and corrected in the second printing (August 2001) are listed separately beginning on page 9 of this file. Errors found after that time are listed immediately below. Please check both lists if you are using the first printing.

Page vii, line -2. Change “a” to “the”: “. . . passes **the** stationary observer.”

Page 6, line +8. “ (t, x, y) ” should be “ $(t, **y**, **z**)$ ”.

Page 14. Exercise 1. The coordinate expression for the first event is missing its closing parenthesis. It should read “ $(1, 0)$.”

Page 21. Exercise 2 (b). The second equality in the second sentence is incorrect and must be deleted. The corrected sentence should read “Show that $F_v = S_v \circ C_v$, where S_v is the Galilean shear of velocity v .”

Page 47. Exercise 4. The subscript of the second matrix should be u_2 , not u_1 . The beginning of the question should read “Show that $H_{u_1} \cdot H_{**u_2**} = \dots$ ”

Page 55, Theorem 2.3. The given proof works only if u (and hence $\sinh u$) is nonnegative. Enlarge the argument as follows, beginning on line -3:

“Therefore, **when $u \geq 0$ and thus $\sinh u \geq 0$,**

$$t = \tau \cosh u + \zeta \sinh u \geq \tau \cosh u - \tau \sinh u = \tau e^{-u} > 0.$$

When $u < 0$ and $\sinh u < 0$, the condition $\zeta < \tau$ implies $\zeta \sinh u > \tau \sinh u$ and so

$$t = \tau \cosh u + \zeta \sinh u > \tau \cosh u + \tau \sinh u = \tau e^u > 0.$$

The three other results are obtained in a similar way. END OF PROOF”

Page 56, line +1. Reverse “two” and “the”; Definition 2.4 should read “*The separation between the two events . . .*”

Page 65, Definition 2.6. Add the following sentence at the end of the definition. “That is, $K_u(X) = -X$ for every vector X in λ_u^\perp .”

Page 67, line +7. Change “circles” to “hyperbolas”: “. . . timelike and spacelike unit hyperbolas. . .”.

Page 70, line +5. The second occurrence of “ LE_2 ” should be “ LE_1 ”:

$$M(E_2) - M(E_1) = LE_2 + C - (LE_1 + C) = L(E_2 - E_1),$$

Page 73, Exercise 14. The condition $\lambda_1 \leq \lambda_2$ should be, instead, $|\lambda_1| \leq |\lambda_2|$. Furthermore, the semimajor and semiminor axes of the ellipse should be $|\lambda_2|r$ and $|\lambda_1|r$.

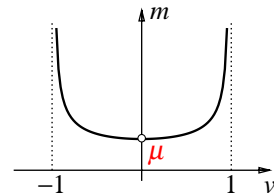
Page 89, line +7. One of the derivatives is missing a “ d ”:

$$\frac{d(\mathbf{p}_1 + \mathbf{p}_2)}{dt} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \mathbf{f}_1 + \mathbf{f}_2 = \mathbf{0},$$

Page 90, line +19. Replace the words “The velocity limitation If this were so,” with “If inertial mass were constant,”

Page 91, line +9. Change “ m ” to “ M ” in the formula $(k/M)T + c/M = 1$.

Page 91, the figure and line –13. Change the two occurrences of m_0 to μ , to agree with the labelling of G ’s rest mass on subsequent pages. Line –13 should read “Nonetheless, it is helpful to single out the value $\mu = m(0)$, . . .”.



Page 97, line +12. Change the first “is” to “it”: “What it says. . .”

Page 97, lines -14 to -11. Replace the lines with:

“While 4-velocity \mathbb{V} is *not* covariant in this sense—the Minkowski norm of \mathbb{V} is different in different frames, but a Lorentz transformation B_w preserves the norm—the *unit* 4-velocity $\mathbb{U} = \mathbb{V}/\|\mathbb{V}\|$ overcomes this problem and is covariant. We can thus think of the components of \mathbb{U} and $\mathbb{P} = \mu\mathbb{U}$ as covariant themselves, in the sense that, if we know the value of all of them ...”

Page 99, line -6. Since 60 mph is *exactly* equal to 88 feet/sec, the first “ \approx ” should be “=”:

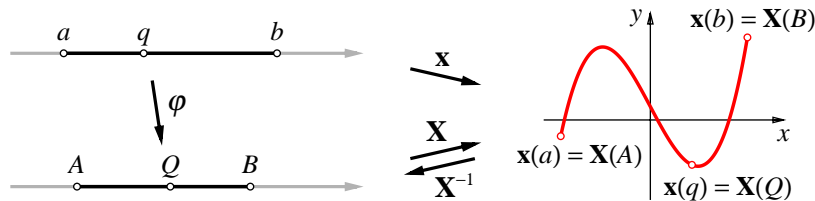
$$60 \text{ mph} = 88 \text{ feet/sec} \approx \dots$$

Page 105, last displayed equation. Add a factor “ μ ”:

$$= \underbrace{\mu c^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}\mu v^2 + \frac{3}{8}\mu \frac{v^4}{c^2} + \dots}_{\text{kinetic energy}} .$$

Page 107, Exercise 6(c). Replace the words “directly from” by “**by assuming H_u is orthochronous and satisfies**”.

Page 112, figure at top of page. The curve on the right should have no self-intersections, because it is meant to illustrate the theorem, which assumes the curve has one-to-one parametrizations. The following is an example of a curve that fits the conditions.



Page 113, line +14. In the middle of the formula for $r(\Delta P)$, change $\mathbf{X}^{-1}(\Delta P)$ to $\mathbf{X}^{-1}(P)$. The formula should read

$$r(\Delta P) = \mathbf{X}^{-1}(P + \Delta P) - \mathbf{X}^{-1}(P) - \nabla \mathbf{X}^{-1}(P) \cdot \Delta P$$

Page 114, line +8. Change $R(\Delta)$ to $R(\Delta Q)$. The beginning of the displayed equations should read

$$\frac{\|\Delta P\|}{|\Delta Q|} = \frac{\|\mathbf{X}'(Q)\Delta Q + R(\Delta Q)\|}{|\Delta Q|} = \left\| \mathbf{X}'(Q) \frac{\Delta Q}{|\Delta Q|} + \frac{R(\Delta Q)}{|\Delta Q|} \right\| = \dots$$

Page 120, line +2. Insert before “Thus . . .” the following sentence in parentheses: “(Note that $\varphi'(s) = -\kappa$ is possible, but this also leads to a circle of radius ρ .)”

Page 128, line -7. Add an “s” to “take”: “. . . a full cycle takes . . .”.

Page 134. line +4. The formula for $\|\mathbb{A}\|$ needs to have the exponent $3/2$ added to the denominator. The formula should read

$$\|\mathbb{A}\| = k|\cos t|/(1 - k^2 \sin^2 t)^{3/2}$$

Page 140. Exercise 5. The first integral in the displayed formula should contain the *derivative* of \mathbb{X} :

$$l = \int_a^b \|\mathbb{X}'(q)\| dq = \int_a^b \sqrt{\left(\frac{dz}{dq}\right)^2 - \left(\frac{dt}{dq}\right)^2} dq.$$

Page 163, line +14. Replace $1/\alpha$ and $e^{\alpha h}/\alpha$ by their reciprocals. The line should read “. . . the accelerations of G and C are α and $\alpha/e^{\alpha h}$, respectively.”

Page 176, line +2. “ $\mathbf{T}(0, 0, \varepsilon)$ ” should be “ $\mathbf{T}(\varepsilon, 0, 0)$.”

Page 186, line -9. There should be no dot product in the displayed equation; it should read “ $\mathbf{A} = -\nabla\Phi$ ”.

Page 211. At the end of the displayed formula above the figure, replace v by \mathbf{v} : “. . . if $\mathbf{v} = v^i \mathbf{x}_i$.”

Page 212, line +12. Insert a “+” between c^2 and Δq^2 : “ $(c^1 + \Delta q^1, c^2 + \Delta q^2)$.”

Page 213, lines +4, +5. These lines give a “little Oh” condition that is different from the “big Oh” condition on line +9. For consistency, change lines 4–5 to agree with the later statement by eliminating the statement that the value of the limit is zero. Instead, the lines should read: “The technical condition is **that**

$$\lim_{(\Delta q^1, \Delta q^2) \rightarrow (0,0)} \frac{\|\mathbf{x}(c^1 + \Delta q^1, c^2 + \Delta q^2) - (\mathbf{x} + \Delta q^1 \mathbf{x}_1 + \Delta q^2 \mathbf{x}_2)\|}{\|(\Delta q^1, \Delta q^2)\|^2}$$

exists and is finite.”

Page 218, Exercise 4. Change the superscript in “ v^2 ” to a subscript: $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Page 252, line +8. The signs of all four entries in the matrix \tilde{B} must be reversed.

Page 258, line -2. The word “and” within the quotes should be “an”: “raising or lowering **an** index.”

Page 261, line -4. The Christoffel symbol $\Gamma_{jk,l}$ should be $\Gamma_{jk,h}$.

Page 266, Exercise 2. In the second equation, the subscript l should be a j : $b_j^i = g^{ik}b_{kj}$.

Page 271, line -4. A differentiation prime is missing; the expression should read “ $\mathbf{Q}'(u) = \mathbf{q}'(\varphi(u))\varphi'(u) = \mathbf{q}'(t)\varphi'(u)$.”

Page 285. All occurrences of c^1 in the large displayed formula should be deleted (to reflect the hyperbolic metric correctly). The final result is unchanged, but the correct formula should read

$$\begin{aligned} \theta &= \arccos \frac{\mathbf{q}' \cdot \mathbf{r}'}{\|\mathbf{q}'\| \|\mathbf{r}'\|} = \arccos \frac{\frac{\dot{q}^1 \dot{r}^1 + \dot{q}^2 \dot{r}^2}{(c^2)^2}}{\sqrt{\frac{(\dot{q}^1)^2 + (\dot{q}^2)^2}{(c^2)^2}} \sqrt{\frac{(\dot{r}^1)^2 + (\dot{r}^2)^2}{(c^2)^2}}} \\ &= \arccos \frac{\dot{q}^1 \dot{r}^1 + \dot{q}^2 \dot{r}^2}{\sqrt{(\dot{q}^1)^2 + (\dot{q}^2)^2} \sqrt{(\dot{r}^1)^2 + (\dot{r}^2)^2}}. \end{aligned}$$

Page 289, line -1. Delete the repeated “the”: “. . . the eigenvectors lie on **the** axes of those coordinates.”

Page 292, Exercise 7a. The asterisk in X_i^* should be deleted; the expression should read $(X_i)^t G X_j = 0$.

Page 297, line +6. The superscript for f should run from 1 to 4:

$$\frac{\partial \varphi^j}{\partial x^1} \frac{\partial f^1}{\partial \xi^i} + \frac{\partial \varphi^j}{\partial x^2} \frac{\partial f^2}{\partial \xi^i} + \frac{\partial \varphi^j}{\partial x^3} \frac{\partial f^3}{\partial \xi^i} + \frac{\partial \varphi^j}{\partial x^4} \frac{\partial f^4}{\partial \xi^i} = \frac{\partial \xi^j}{\partial \xi^i} = \delta_i^j.$$

Page 297, line –6. Change the superscript of ξ from i to j in one place:

$$x^k = x^k(\xi^i), \quad \xi^j = \xi^j(x^l),$$

Page 306, Exercise 6. In both parts of the question change $\alpha = 1$ to $\alpha = 0.1$.

Page 308, line –10. Add the following sentence at the end of the paragraph.

“Note that *all* tensors—including those with positive contravariant rank—are “generally covariant” when we use the term in the general sense.”

Page 313, line –2. The superscripts for the three appearances of “ ξ ” in the last term must be permuted, as follows:

$$\dots = \left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right) \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \frac{\partial x^r}{\partial \xi^k} + 2g_{pq} \frac{\partial^2 x^p}{\partial \xi^i \partial \xi^j} \frac{\partial x^q}{\partial \xi^k}$$

Page 316, In the third displayed formula, the superscript “ h ” must be replaced by “ k ”:

$$\alpha^j \Gamma_{ij}^k = \underbrace{\alpha^j \Gamma_{pq}^r}_{\text{}} \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \frac{\partial \xi^k}{\partial x^r} + \alpha^j \frac{\partial^2 x^p}{\partial \xi^i \partial \xi^j} \frac{\partial \xi^k}{\partial x^p}$$

Page 324, line +6. The factor B^j in the second term of the displayed equation needs the subscript h . The equation should read

$$\frac{dB_h^l}{dt} + \Gamma_{jm}^l B_h^j \frac{dy_k^m}{dt} = 0.$$

Page 325, line –10. The expression on the left-hand side of the displayed equation needs the appropriate parallel transport map added, as stipulated in the sentence preceding the one containing this equation:

$$\frac{\tau_{\mathbf{y}_k(0), \mathbf{y}_k(h)}^{-1}(\mathbf{a}(\mathbf{y}_k(h))) - \mathbf{a}(\mathbf{y}_k(0))}{h} = \dots$$

Page 325, line –1. The same kind of correction to the displayed equation is needed as in line –10, above:

$$\lim_{h \rightarrow 0} = \frac{\tau_{\mathbf{y}_k(0), \mathbf{y}_k(h)}^{-1}(\mathbf{T}(\mathbf{y}_k(h))) - \mathbf{T}(\mathbf{y}_k(0))}{h} = T_{J;k}^I.$$

Page 327, Exercise 10. The indices in the formula do not match unless the index-raising factor “ g^{ij} ” is added on the left; parentheses also help clarify what is intended. Thus, the formula should read $g^{ij} (T_{J,j;l}^I) = (T_J^{I,i})_{;l}$.

Page 327, Exercise 12a. The superscript “ i ” in the second term should be “ h ”: $a_{;j;k}^h - a_{;k;j}^h = -R_{ijk}^h a^i$.

Page 331, line +11. Add the words “freely falling”: “But the worldcurve of every **freely falling** particle is a straight line. . . .”

Page 335, In the fourth displayed formula, the components of Γ_P^{-1} should have superscripts, not subscripts:

$$\Gamma_P^{-1} = \begin{pmatrix} \gamma^{00} & \gamma^{03} \\ \gamma^{30} & \gamma^{33} \end{pmatrix} = \dots$$

Page 341, Exercise 1d. The formula for ρ is missing a term. The formula should read

$$\rho = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) - \nabla^2 A^0.$$

Page 344, Exercise 12. Insert the phrase “when $a^2 b^3 - a^3 b^2 > 0$ ” in the first sentence: “. . . and show that it turns to the right **when $a^2 b^3 - a^3 b^2 > 0$** , in the sense. . . .”. Delete the word “all” from the second sentence: “. . . and causes geodesics to turn. . . .”.

Page 350, In the last sentence of the statement of Corollary 7.1, replace “is” by “satisfies the equations”: “Then the rate of separation of geodesics along $z^k(\tau)$ **satisfies the equations**”

$$\frac{D^2}{d\tau^2} \frac{\partial x^h}{\partial q} = -K_j^h \frac{\partial x^j}{\partial q}, \quad \text{where} \quad K_j^h = R_{ijk}^h \frac{dz^i}{d\tau} \frac{dz^k}{d\tau}.$$

Page 351, lines +6, +7. Add “a” in two places: “ K_j^h is **a** 4×4 matrix, not **a** 3×3 .”

Page 355, line +9. Add “to”: “. . . when we use it **to** determine . . .”.

Page 365, Exercise 10 (c). For consistency with part (d), replace the three occurrences of lower case “s” in the formula with upper case “S”:

$$\eta_{T,n}(S) = \left(\frac{1}{\alpha} \tanh^{-1} \left(\frac{T}{S + 1/\alpha} \right), \frac{1}{2\alpha} \ln(\alpha^2((S + 1/\alpha)^2 - T^2)) \right)$$

Page 368, line −10. Insert “... are covariant in the general sense (cf. page 308); they are tensors ...”.

Page 378, line +9. Delete the words “dimensionally homogeneous”.

Page 382, Exercise 7 (b). In the last word, change “α” to “β”: “... the flow of energy in the β-direction.”

Page 385, lines −5, −4. Change from present to past tense for consistency: “... he derived a relativistic gravitational field that he used to calculate ...”.

Page 386, line −4. Insert “... boost $B_v : G \rightarrow R$ (cf. Exercise 5, §3.2):”

Page 393, line −6. The second occurrence of “α” should be “β”:

$$\dots = O\left(\frac{1}{c^2}\right) \left(c + O\left(\frac{1}{c}\right)\right) \left(\dot{x}^\beta + O\left(\frac{1}{c^2}\right)\right) = \dots$$

Page 412, Exercise 11 (a). In the formula for the scalar curvature R , the terms T'/r and Q'/r need to be doubled:

$$R = R_i^i = -\frac{2}{r^2} + e^{-Q} \left(T'' + \frac{(T')^2}{2} - \frac{T'Q'}{2} + \frac{2T'}{r} - \frac{2Q'}{r} + \frac{2}{r^2} \right).$$

Page 417, line +11. Change the colon to a semicolon and add the following: “... curvature; the following are equal up to order $O(1/c^2)$.”

Page 425, lines +13, +15. We should use the norms of $\mathbf{p} \times \mathbf{q}$ and $\mathbf{x} \times \mathbf{v}$: “ $(\mathbf{p} \cdot \mathbf{q})^2 + \|\mathbf{p} \times \mathbf{q}\|^2 = p^2 q^2$ and

$$(\mathbf{x} \cdot \mathbf{v})^2 + \|\mathbf{x} \times \mathbf{v}\|^2 = (r\dot{r})^2 + J^2 = r^2 v^2,”$$

Page 429. A factor e^2 is missing from terms in the third and the fifth displayed equations. The third equation should read

$$\frac{d^2 v}{d\theta^2} + v = \frac{3\mu^3}{J^4} \left(1 + \frac{e^2}{2}\right) + \frac{6e\mu^3}{J^4} \cos \theta + \frac{3e^2\mu^3}{2J^4} \cos 2\theta + O\left(\frac{1}{c^2}\right).$$

The fifth equation should read

$$v_A = \frac{3\mu^3}{J^4} \left(1 + \frac{e^2}{2}\right), \quad v_B = \frac{3e\mu^3}{J^4} \theta \sin \theta, \quad v_C = -\frac{e^2\mu^3}{2J^4} \cos 2\theta.$$

Page 433, Exercise 6 (b). The error on page 429 is repeated here. The expression C needs to have the factor e^2 added:

$$C = \frac{3e^2\mu^3}{2J^4} \cos 2\theta.$$

Page 442, under the index item *divergence*: the *divergence of a tensor* is defined on page 373, not 372. There are other index items with page ranges that should end on 373, not 372.

The following errors in the first edition have been corrected in the second printing (August 2001).

Page 27. The displayed formulas for τ and for E at the top of the page are wrong. They should be

$$\tau = \frac{t - vz}{\sqrt{1 - v^2}},$$
$$E(t, z) = \mathcal{E} \left(\frac{t - vz}{\sqrt{1 - v^2}}, \frac{z - vt}{\sqrt{1 - v^2}} \right).$$

Page 28. Exercise 5. The formula for $E(t, z)$ must be changed as on page 27.

Page 89, line +8. Add the word "does": "the total momentum **does** not change over time."

Page 137. Replace all the text from Proposition 3.4 up to, but not including, Definition 3.7 by the following.

It is evident that the left-hand side—and thus the right—has the dimensions of a rate of change of energy with respect to time. The following proposition will lead us to a physical interpretation for $\mathbf{f} \cdot \mathbf{v}$.

Proposition 3.4 If $\tilde{K}(t) = \frac{1}{2}\mu v^2(t)$ is the classical kinetic energy of G in R 's frame and $\tilde{\mathbf{f}} = \mu\mathbf{a}$ is the classical 3-force acting on G , then

$$\frac{d\tilde{K}}{dt} = \tilde{\mathbf{f}} \cdot \mathbf{v}.$$

PROOF: Since $v^2 = \mathbf{v} \cdot \mathbf{v}$, we can write $\tilde{K}(t) = \frac{\mu}{2}\mathbf{v} \cdot \mathbf{v}$. Therefore,

$$\frac{d\tilde{K}}{dt} = \frac{\mu}{2}(\mathbf{v} \cdot \mathbf{v})' = \frac{\mu}{2}(2\mathbf{v}' \cdot \mathbf{v}) = \mu\mathbf{v}' \cdot \mathbf{v} = \mu\mathbf{a} \cdot \mathbf{v} = \tilde{\mathbf{f}} \cdot \mathbf{v}. \quad \text{END OF PROOF}$$

Let us therefore interpret $\mathbf{f} \cdot \mathbf{v}$ (which involves the *relativistic* 3-force \mathbf{f}) as the time rate of change of *relativistic* kinetic energy K of G in R 's frame. Thus $\mathbb{F} \cdot \mathbb{U} = 0$ becomes

$$c^2 \frac{dm}{dt} = \frac{dK}{dt}, \quad \text{implying} \quad c^2 m = K + \text{const.}$$

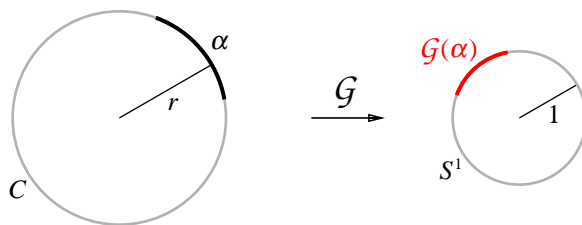
If we further require that $K = 0$ when $v = 0$, as in the classical case, then we can determine the constant of integration in the last equation: Since $m = \mu$ and $K = 0$ when $v = 0$, it follows that $\text{const} = \mu c^2$, the rest energy of G . We can summarize the previous discussion in the following definition and corollary.

Page 138, line 4. Insert between “ K ” and the colon:

“We also have an expression for the relativistic kinetic energy K that agrees with the one we used in section 3.1 for non-accelerated motion:”

Page 190, line -4. Insert the sentence: “... $hN = h\nu - h\nu\Delta\Phi$. (For the rest of this paragraph, h represents Planck’s constant, not the position of C .) If we let...”.

Page 244. In the figure, the Gauss image $\mathcal{G}(\alpha)$ in S^1 is incorrect; it should be rotated 90° counterclockwise, as in the figure below.



Page 268. The statement about the normal component \mathbf{N} of acceleration (and the formula given in line -4) are incorrect, as is the displayed formula immediately above that line. The displayed formula and the sentence following it should read

$$\left(\frac{d^2 q^k}{dt^2} + \Gamma_{ij}^k \frac{dq^i}{dt} \frac{dq^j}{dt} \right) \mathbf{x}_k + b_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} \mathbf{n}.$$

The last term is the normal component \mathbf{N} of acceleration. It depends on the second fundamental form b_{ij} and the velocity components dq^i/dt of the curve $\mathbf{z}(t)$, but is in general different from zero.

Page 375, line -1. Add a space between “ δ_k^h ” and “is”.

Page 382. In exercise 9 (c), two minus signs are missing. The statement should read:

Assume $T_{ij} \equiv 0$ and show that $R = -4\Lambda$ where R is the scalar curvature function $R = R_i^i$. Then show that $R_{ij} = -\Lambda g_{ij}$.

Page 431, The paragraph “Visualizing the drift” states incorrectly the size of the drift due to the other planets. Replace the paragraph with the following.

In fact, most of the shift is due to the fact that the observations are made in a non-inertial frame; only about one-tenth is due to the Newtonian gravitational “disturbances by other planets.” The relativistic contribution is extremely small; even over 80 centuries it amounts to just under 1° , but it is enough to be visible in the figure on the next page. By contrast, the observed shift during the same time is enormous—a third of a complete revolution. The figure is drawn to scale and gives an accurate picture of the eccentricity of Mercury’s orbit. The non-relativistic contribution (of about $5557''$ per century) is shown by the dashed ellipse, the correct (relativistic) drift by the gray ellipse.